Idempotents of Semigroups

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Contents

1 Introduction

This essay is about the structure of semigroups, and the extent to which this structure is encoded in the properties of their idempotents. The information about a semigroup that can be recovered by considering only its idempotents ranges from nothing (in the case of groups), through to a complete reconstruction of the semigroup multiplication table.

Green's relations restricted to idempotents suggest two naturally occurring pre-orders on the idempotents of a semigroup, with respect to which they form a biordered set. In the case of inverse semigroups (the most widely studied class of semigroups after groups), the idempotents commute and the biordered set becomes a partially ordered set in which any two elements have a greatest lower bound — that is, a lower semilattice. The abstract notion of a biordered set was introduced in the 1970s by Nambooripad [24, 25] in his description of the partial algebras of idempotents of regular semigroups. Nambooripad's technique relied on his regularity axiom, which requires that all sandwich sets be non-empty.

Easdown [12] showed in the 1980s that, by relaxing the regularity axiom, abstract biordered sets characterise the partial algebras of idempotents of arbitrary semigroups. One of the main tools in Easdown's proof is a representation of a biordered set by transformation and dual transformation semigroups. This representation is introduced and placed in the context of other naturally occurring semigroup representations in the work of Easdown and Hall [15], and exploited by Easdown [10, 11, 13] to classify the biordered sets of several important classes of semigroups. It was used recently by McElwee [22], who develops his own calculus for dealing with biordered sets in which all the principal ideals are partially ordered sets.

This essay aims to convince the reader of the power of the biordered set approach to the study of semigroups, and to present some of the principal tools by which these classifications of biordered sets can be effected. The first half of this essay explores the mathematical objects crucial to semigroup theory, proves some important results needed later on (though nonetheless interesting in their own right) and introduces several of the examples which feature as running themes throughout. The second half, beginning at Section 6, covers the main classes of semigroups in order of increasing generality, and their characterisation in terms of idempotent properties. In Section 8, the biordered set representation referred to above is introduced. This is then used in later sections in the sketches of proofs, most notably the characterisation of biordered sets of finite semigroups in Section 11, which also uses a novel application of Ramsey's Theorem from graph theory, as well as to illustrate the reconstruction of semigroups from particular examples of biordered sets. The characterisations in Sections 7 and 9 of biordered sets of the classes of regular semigroups, unions of groups, and bands are paralleled in Section 10 by the characterisations for the classes of eventually regular, group-bound and periodic semigroups respectively.

Wherever possible the original sources for the results or techniques concerning biordered sets are indicated in the text. For general theory and facts about semigroups the sources are [19, 20], and sketches or notes provided by my supervisor. For reasons of space, the most difficult results are sometimes only quoted or sketched. The examples here have been chosen so that all of the main ideas in the theorems can be illustrated and readily understood by the reader.

1.1 Basic ideas and definitions

This section introduces the basic concepts needed to study semigroups. A reader familiar with these definitions and preliminary results may choose to skim the chapter, noting the conventions outlined in section 1.2.

A *semigroup* is any set with an associative binary operation, usually referred to as multiplication, which we denote by juxtaposition. Associativity yields Cayley's Theorem, which says that any semigroup is isomorphic to a semigroup of full transformations of a set under the operation of composition. We consider several important classes of semigroups, some of which (such as groups) may already be familiar to the reader.

A *monoid* is a semigroup with an identity element 1. Even though a semigroup will often not have a 1, it is still possible to have inverses. If S is a semigroup, we say that s' is an *inverse* (not necessarily unique) of the element s if both $s = ss's$ and $s' = s'ss'$. Notice that this reduces to the more familiar notion of inverse in the case where S is a group. Any element with an inverse is called *regular*, and if all elements of S have inverses, we say that S is a *regular semigroup*[1]. If every element has a unique inverse, S is called an *inverse semigroup.* Note that if $x \in S$ and $x = xax$ for some $a \in S$ then x is regular, because $x(axa)x = x$ and $(axa)x(axa) = axa$, so that axa is an inverse of x.

Semigroups may have *subsemigroups*, that is, non-empty subsets which are closed under the semigroup operation, and *subgroups*, that is, subsemigroups which are groups under the inherited operation. If S is a monoid then the subset $\{x \in S \mid xy = yx = 1, \exists y \in S\}$ is the maximal subgroup of S containing 1, referred to as the *group of units* of S. If S is a union of its subgroups, we call it a *union of groups.* Later we see that this union may be taken to be disjoint and each subgroup maximal.

An *idempotent* is any element e such that $e^2 = e$. Idempotents are their own inverses, as can be seen immediately from the definition of inverse. When an idempotent is conjugated by an element of the group of units, the result is another idempotent: $(g^{-1}eg)(g^{-1}eg) = g^{-1}eg$, for any idempotent e and any element g in the group of units.

If a semigroup consists entirely of idempotents, it is called a *band.* It is possible to form a semigroup from two non-empty sets X and Y by defining an associative multiplication on the product $X \times Y$ with the rule $(x, y)(x', y') = (x, y')$, for $x, x' \in X$, $y, y' \in Y$, with respect to which all elements are idempotent. We call the band $(X \times Y, \cdot)$ *rectangular.* A *left [right] zero semigroup* is any non-empty set S with multiplication given by $xy = x$ $[yx = x]$ for any $x, y \in S$. This is isomorphic to a specific instance of a rectangular band, as can be seen by putting $X = S$ $[Y = S]$ and letting Y $[X]$ be any singleton set in the definition above. Conversely any rectangular band is easily seen to be isomorphic to a direct product of a left zero with a right zero semigroup.

The *dual* S^* of a semigroup S is defined to be the semigroup

$$S^* = \{x^* \mid x \in S\},$$

with multiplication $x^*y^* = (yx)^*$. For example, right zero semigroups are the duals of left zero semigroups. We will frequently appeal to duality to prove things where we would otherwise have to give two very similar arguments.

[1]This notion of regularity comes from the use of the term in ring theory, where a ring is *(Von Neumann) regular* if its multiplicative semigroup is regular in the sense just defined.

1.2 Conventions

Where functions are used in this essay, the algebraic left-to-right notation has been employed. If the composition of two (full or partial) functions ϕ and ψ is written as $\phi \circ \psi$ (or $\phi\psi$), the intention is to apply ϕ and then ψ. We use $x\phi$ to indicate ϕ applied to x.

Let S be a semigroup. Define the product of subsets T, U of S to be

$$TU = \{tu \mid t \in T, u \in U\},$$

and for the purposes of this operation on sets, identify any singleton set $\{a\}$ with its element a. Note that this operation is associative. If X is a subset of S then we write $\langle X \rangle$ for the subsemigroup of S *generated by* X, that is, the smallest subsemigroup of S containing X. We write $E(S)$ to refer to the set of idempotents of S. We use $\mathrm{Mat}_n(F)$ to denote the set of $n \times n$ matrices with entries taken from the set F. The set of natural numbers $\{1, 2, \ldots n\}$ is written $[n]$.

It is often useful to use $S^1 = S \cup \{1\}$, where 1 is a new element, and we extend the multiplication of S to S^1 by requiring that $x1 = 1x = x$ for all $x \in S^1$. If $a \in S$, we call $Sa \cup \{a\}$ the *principal left ideal generated by* a, which we can also write S^1a. The *principal right ideal generated by* a is defined dually. In the case of commutative semigroups, there is no distinction between left and right principal ideals, so we can refer to them simply as *principal ideals*. Note that if S is a monoid or regular, then $S^1a = Sa$.

The symbol $\Box$ is used in this essay to mark the end of a proof or an example.

2 Partially Ordered Sets

Relations, pre-orders and partial orders play a central role in this essay. In this section, we look at the important classes of relations that will be needed, and prove the equivalence of semilattices and semigroups of commuting idempotents.

A *relation* on a set X is any subset of $X \times X$. If $\rho \subseteq X \times X$ we may write $x\rho y$ when $(x, y) \in \rho$. We denote the *identity* relation $\{(x, x) \mid x \in X\}$ by 1_X, and the *universal* relation $X \times X$ by ω_X.

Let ρ, τ be relations on a set X. Then we define

$$\rho^{-1} \quad = \quad \{(y, x) \in X \times X \mid (x, y) \in \rho\}$$

and

$$\rho \circ \tau \quad = \quad \{(x, z) \in X \times X \mid (x, y) \in \rho,\ (y, z) \in \tau\ \ (\exists y \in X)\}.$$

Notice that the binary operation $\circ$ on relations is associative, so bracketing can be omitted. We put $\rho^n = \underbrace{\rho \circ \rho \circ \rho \circ \cdots \circ \rho}_{n\ copies}$.

We say that ρ, τ *commute* if $\rho \circ \tau = \tau \circ \rho$. We call ρ *reflexive* if $1_X \subseteq \rho$, *symmetric* if $\rho^{-1} = \rho$, *transitive* if $\rho \circ \rho \subseteq \rho$, and an *equivalence relation* (or just *equivalence*) if it is reflexive, symmetric and transitive.

If ρ is reflexive and transitive, it is called a *pre-order*. We call ρ *antisymmetric* if $\rho \cap \rho^{-1} = 1_X$, and a *partial order* if it is reflexive, transitive and antisymmetric. Clearly all partial orders are pre-orders. We often use the symbol $\leq$ to denote a partial order. If ρ is a partial order and $\rho \cup \rho^{-1} = \omega_X$, we call ρ a *total order* and X a *chain*. Partially ordered sets can be represented by *Hasse diagrams*, in which the elements are connected with line segments to describe the partial order completely. For any elements x, y in the set, a line is drawn upwards from x to y if $x \leq y$ and there is no other distinct element z such that $x \leq z \leq y$.

2.1 Semilattices: semigroups of commuting idempotents

If $(X, \leq)$ is a partially ordered set, and $Y \subseteq X$, we say that $a \in Y$ is *minimal* if for all $y \in Y$, $y \leq a \Rightarrow y = a$. If $a \leq y$ for all $y \in Y$, we say a is *minimum*. *Maximal* and *maximum* are defined as would be expected. A *lower bound* for Y is any element $b \in X$ such that $b \leq y$ for all $y \in Y$. If the set of lower bounds of Y has a maximum element c, it is called the *greatest lower bound* of Y, which must be unique, and when $Y = \{a, b\}$, we write $c = a \wedge b = b \wedge a$. If $(X, \leq)$ is such that $a \wedge b$ exists for every a, b in X, we say that $(X, \leq)$ is a *(lower) semilattice.*

Proposition 2.1. Let $(E, \leq)$ be a semilattice. Then $(E, \wedge)$ is a semigroup of commuting idempotents, and for all $a, b \in E$, $a \leq b$ if and only if $a \wedge b = a$.

Proof. The operation $\wedge$ is clearly commutative, and $a \wedge a = a$ for all $a \in E$. Let $a, b, c \in E$ and put $d = (a \wedge b) \wedge c$. Then $d \leq a \wedge b \leq a, b$ and $d \leq c$, so d is a lower bound for $\{a, b, c\}$. If e is also a lower bound for $\{a, b, c\}$ then in particular e is a lower bound for $\{a, b\}$, so $e \leq a \wedge b$, and also $e \leq c$, so $e \leq (a \wedge b) \wedge c = d$. Thus d is the greatest lower bound for $\{a, b, c\}$. Similarly, $a \wedge (b \wedge c)$ is also the greatest lower bound for $\{a, b, c\}$, so

$$(a \wedge b) \wedge c \;=\; a \wedge (b \wedge c),$$

proving $\wedge$ is associative. Thus $(E, \wedge)$ is a semigroup of commuting idempotents. □

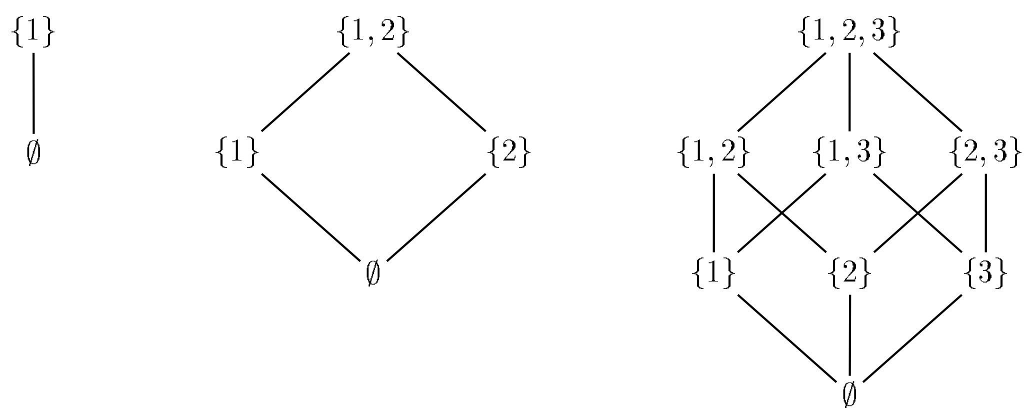
Proposition 2.2. Let E be a semigroup of commuting idempotents. Then E is a semilattice with respect to the relation given by $a \leq b \Leftrightarrow ab = a$. For any $a, b \in E$, we have $a \wedge b = ab$.

Proof. We have $a = a^2$, so $a \leq a$, showing reflexivity. If $a \leq b$ and $b \leq a$ then $a = ab = ba = b$, and we have antisymmetry. Finally, if $a \leq b$ and $b \leq c$, then $ab = a$, $bc = b$ so $ac = abc = ab = a$, whence $a \leq c$, proving transitivity. This shows that $\leq$ is a partial order.

For any $a, b \in E$, $a(ab) = ab$, so $ab \leq a$ and $b(ab) = (ab)b = ab$, so $ab \leq b$. Thus ab is a lower bound for $\{a, b\}$. If $c \leq a$ and $c \leq b$ then $c(ab) = cb = c$, giving $c \leq ab$, from which we obtain $a \wedge b$ exists and equals ab. Hence $(E, \leq)$ is a lower semilattice. □

Propositions 2.1 and 2.2 above allow us to regard lower semilattices and semigroups of commuting idempotents as equivalent concepts. The *Munn inverse semigroup* T_E of E is defined to be the set of all isomorphisms between the principal ideals of a semilattice E [23]. It is easy to see that each principal ideal is the set of all elements below or equal to a given element of the semilattice. Then T_E becomes a semigroup (in fact a subsemigroup of the *symmetric inverse semigroup* on E, discussed below) under the operation of composition of partial mappings. The importance of T_E is derived from the fact that it is the maximum fundamental inverse semigroup whose idempotents form a semilattice isomorphic to E. The semilattice isomorphism takes $e \in E$ to the identity mapping on the principal ideal Ee. Inverse semigroups are discussed in more detail in Section 6. The definition of fundamental and the general theory of T_E are not required in this essay.[2] However, illustrations of the calculation of T_E are given in the next two examples.

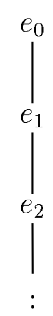
Example 2.3. Let E be the set of subsets of a set X. Then E is partially ordered by set containment, and the greatest lower bound of any two subsets under this partial order is their intersection, so that E becomes a semilattice. The following Hasse diagrams depict the semilattices of subsets of [1], [2] and [3], respectively.



[2] A good exposition of them can be found in [20] or [19].

It turns out that T_E for each of these semilattices is isomorphic to the *symmetric inverse semigroup* $\mathcal{I}(X)$ of partial injective mappings from $X \to X$. The key point is that any isomorphism of principal ideals of E must fix the atoms of the semilattice, that is, the singleton subsets of X, and is uniquely determined by its action on these atoms. Thus we have a correspondence between T_E and $\mathcal{I}(X)$ which can easily be seen to preserve the operation of composition of partial mappings, yielding a semigroup isomorphism. The symmetric inverse semigroup is discussed in more detail in Section 5. There we will see that the idempotents of $\mathcal{I}(X)$ form a semilattice in a natural way which can be identified with E. This is an example of one of the themes of this essay — how information stored in the relationships between idempotents can be used to recover information (sometimes complete) about the semigroup from which they come. □

Example 2.4. Let $E = \{e_0, e_1, e_2, \dots\}$ be a set in one-one correspondence with the natural numbers, which becomes partially ordered with respect to $\leq$ defined by $e_j \leq e_i$ if and only if $i \leq j$. The Hasse diagram of E is an infinite descending chain with largest element e_0:



Then E is a semilattice, and becomes a semigroup of commuting idempotents under the operation $e_i e_j = e_i \wedge e_j = e_{\max\{i,j\}}$. Observe that each principal ideal of E is isomorphic to E, and that for each i, j, there is a unique isomorphism from Ee_i to Ee_j, namely

$$\phi_{i,j} : \; Ee_i \longrightarrow Ee_j, \qquad e_k \longmapsto e_{k+(j-i)} \quad (k \geq i).$$

Put $x = \phi_{0,1}$ and $y = \phi_{1,0}$. Then, as partial mappings under the operation of composition (wherever possible), $xy = \mathrm{id}_E$, from which it follows quickly by cancellation that $\langle x, y\rangle = \{y^i x^j \mid i, j \geq 0\}$, with the convention that $y^0 = x^0 = \mathrm{id}_E$, and also that

$$E(\langle x, y\rangle) \;=\; \{y^i x^i \mid i \geq 0\}.$$

Note further that, for each $i, j \geq 0$, $\phi_{i,j} = y^i x^j$, so that $T_E = \langle x, y\rangle$, which is known in the literature (see [20]) as the *bicyclic semigroup*. Identifying e_i with ϕ_{ii} for each i, the elements of T_E may be displayed as follows, with the Hasse diagram of the original semilattice appearing down the diagonal:

$$\begin{array}{cccccc}
e_0 & x & x^2 & x^3 & x^4 & \cdots \\
y & e_1 & yx^2 & yx^3 & yx^4 & \cdots \\
y^2 & y^2x & e_2 & y^2x^3 & y^2x^4 & \cdots \\
y^3 & y^3x & y^3x^2 & e_3 & y^3x^4 & \cdots \\
y^4 & y^4x & y^4x^2 & y^4x^3 & e_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Note that the bicyclic semigroup (represented in this way as partial mappings) is induced from the infinite cyclic group generated by the permutations

$$x : \mathbb{Z} \longrightarrow \mathbb{Z},\;\; i \longmapsto i+1 \qquad \text{and} \qquad y : \mathbb{Z} \longrightarrow \mathbb{Z},\;\; i \longmapsto i-1,$$

by restricting the domain of x to $\{z \in \mathbb{Z} \mid z \geq 0\}$ and the domain of y to $\{z \in \mathbb{Z} \mid z \geq 1\}$. Whilst a group has only one idempotent, the bicyclic semigroup has infinitely many, and all of the information about the semigroup can be recovered from the relationships between idempotents using T_E.

$\square$

3 Dually Ordered Sets

Let X be a set and let $\longrightarrow$ and $\succ\!\!-$ be two relations on X, which we refer to as a *right arrow* and *left arrow* respectively. Call $(X, \rightarrow, \succ\!\!-)$ a *dually ordered set* if $\longrightarrow$ and $\succ\!\!-$ are pre-orders, and

$$(\forall x, y \in X) \;\; x \longrightarrow y \text{ and } x \mathrel{-\!\!\prec} y \quad \Rightarrow \quad x = y.$$

Although neither pre-order necessarily has the property of antisymmetry, the preceding property generalises antisymmetry in the sense that if $\longrightarrow \; = \; \succ\!\!-$ then $\longrightarrow$ is antisymmetric and $(X, \rightarrow)$ is a partial order. Conversely, if $(X, \leq)$ is partially ordered then $(X, \rightarrow, \succ\!\!-)$ is dually ordered by taking $\longrightarrow \; = \; \succ\!\!- \; = \; \leq$. The notion of a dually ordered set is illustrated by the following three related examples.

3.1 Illustrations

Example 3.1. Let V be a vector space, and let $X = \{(U, W) \mid U, W$ are subspaces of V and $V = U \oplus W\}$. Define relations

$$\begin{aligned} (U_1, W_1) &\longrightarrow (U_2, W_2) \quad \text{if} \quad U_1 \subseteq U_2, \\ (U_1, W_1) &\succ\!\!- (U_2, W_2) \quad \text{if} \quad W_1 \supseteq W_2. \end{aligned}$$

The relations are both pre-orders, since they inherit reflexivity and transitivity directly from $\subseteq$. We claim that X is dually ordered by $\longrightarrow$ and $\succ\!\!-$.

Say $(U_1, W_1) \longrightarrow (U_2, W_2)$ and $(U_1, W_1) \mathrel{-\!\!\prec} (U_2, W_2)$. Then $U_1 \subseteq U_2$ and $W_1 \subseteq W_2$. We aim to show $U_2 \subseteq U_1$. Take any $u \in U_2$. Observe that $u \in V = U_1 \oplus W_1$, so $u = u_1 + w_1$ for some $u_1 \in U_1$, $w_1 \in W_1$. Hence $w_1 = u - u_1 \in U_2$ since both u, $u_1 \in U_2$. Thus $w_1 \in U_2 \cap W_2 = \{0\}$, and we have $u = u_1 \in U_1$, proving $U_2 \subseteq U_1$, whence $U_2 = U_1$. By a similar argument, we obtain $W_1 = W_2$, thus $(U_1, W_1) = (U_2, W_2)$, showing that $(X, \rightarrow, \succ\!\!-)$ is a dually ordered set. □

Example 3.2. Let S be a semigroup, and let $E = E(S)$ be the set of idempotents of S with relations

$$\begin{aligned} e \longrightarrow f \quad &\text{if} \quad fe = e, \\ e \mathrel{\succ\!\!\!-\!\!\!-} f \quad &\text{if} \quad ef = e. \end{aligned}$$

Reflexivity follows immediately from the fact that the elements are idempotents. If $e \longrightarrow f$ and $f \longrightarrow g$ then $ge = gfe = fe = e$, whence $e \longrightarrow g$ and we have transitivity, proving that $\longrightarrow$ (and similarly $\mathrel{\succ\!\!\!-\!\!\!-}$) is a pre-order. If $e \longrightarrow f$ and $e \mathrel{-\!\!\!-\!\!\!\prec} f$, then $e = fe = f$. Hence $(E, \rightarrow, \mathrel{\succ\!\!\!-})$ is a dually ordered set. $\square$

Example 3.3. Let F be a field, and take any integer $n \geq 1$. Put $S = \mathrm{Mat}_n(F)$, and $E = E(S)$. Matrix multiplication is associative, so S is a semigroup, and E is dually ordered by the relations $\longrightarrow$ and $\mathrel{\succ\!\!\!-\!\!\!-}$ defined in Example 3.2. In fact, we can define a correspondence between E and the dually ordered set of Example 3.1 which preserves the pre-orders, where $V = F^n$. Let $M \in E$. Put $\ker M = \{\mathbf{v} \in F^n \mid M\mathbf{v} = \mathbf{0}\}$, the solution space of M, and $\mathrm{im}\ M = \{M\mathbf{v} \mid \mathbf{v} \in F^n\}$, the column space of M. We regard elements of F^n as column vectors. It is a straightforward exercise to show $F^n = \mathrm{im}\ M \oplus \ker M$. Observe that for $M, N \in E$,

$$M \longrightarrow N \quad \Leftrightarrow \quad NM = M \quad \Leftrightarrow \quad \mathrm{im}\ M \subseteq \mathrm{im}\ N$$

and

$$M \mathrel{\succ\!\!\!-\!\!\!-} N \quad \Leftrightarrow \quad MN = M \quad \Leftrightarrow \quad \ker M \supseteq \ker N.$$

Put $X = \{(U, W) \mid U, W$ are subspaces of F^n and $F^n = U \oplus W\}$. Let $\theta : E \to X$ where $M \mapsto (\mathrm{im}\ M, \ker M)$. Then it is a straightforward exercise to verify that θ is one-one and onto. The preceding observation shows that θ preserves arrows, where for X, arrows are defined as in Example 3.1. $\square$

Observe that $\longleftarrow \; = \; \longrightarrow^{-1}$ and $\mathrel{-\!\!\!-\!\!\!\prec} \; = \; \mathrel{\succ\!\!\!-\!\!\!-}^{-1}$. We then combine arrows in various ways.

$$e \rightarrowtail f \quad \text{if and only if} \quad e \mathrel{\succ\!\!\!-\!\!\!-} f \text{ and } e \longrightarrow f,$$

$$e \longleftrightarrow f \quad \text{if and only if} \quad e \longrightarrow f \text{ and } e \longleftarrow f, \text{ and}$$

$$e \succ\!\!\relbar\!\!\prec f \quad \text{if and only if} \quad e \succ\!\!\relbar\!\!\relbar f \text{ and } e \relbar\!\!\relbar\!\!\prec f.$$

Later these arrows are arranged diagrammatically with obvious meanings. Because of the antisymmetry-like property of dually ordered sets, these are the only possible configurations of two arrows if e and f are taken to be distinct. This highlights the convenience of the arrow notation introduced by Easdown and Hall [9, 15].

We would like to know what properties of a given abstract dually ordered set guarantees that it arises as an instance of Example 3.2. An answer to this might be a step towards constructing semigroups by first building a structure or 'skeleton' involving just idempotents.

4 Axioms for Biordered Sets

In this chapter, we give the axioms defining biordered sets. Although these axioms appear technical at first, they arise by abstracting properties of Green's relations restricted to idempotents. We will look more closely at Green's relations in the next section. The reader will notice that some earlier axioms are needed for the sense of later axioms. All except the first axiom occur in dual pairs, indicated here by an asterisk.

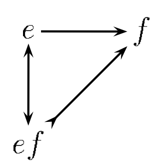
Let E be any set with a partial multiplication (denoted by juxtaposition) over a domain $D_E \subseteq E \times E$, and $\longrightarrow$ and $\succ\!\!\!-\!\!-$ be relations on E. We call $(E, \rightarrow, \succ\!\!-)$, or simply E, a *biordered set* if the following axioms are satisfied.

(B1) $\rightarrow$ and $\succ\!\!-$ are reflexive and transitive, $D_E = \rightarrow \cup \succ\!\!- \cup \leftarrow \cup -\!\!\prec$, and

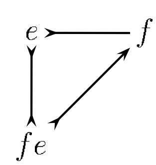
$$e \longrightarrow f \;\Leftrightarrow\; fe = e, \qquad\qquad e \succ\!\!\!-\!\!- f \;\Leftrightarrow\; ef = e.$$

Note that this axiom guarantees that $(E, \rightarrow, \succ\!\!-)$ is dually ordered. It also tells us that if there is a right [left] arrow from e to f then the product of ef $[fe]$ is defined. The next two axioms describe the relationships that arise between e, f and ef $[fe]$.

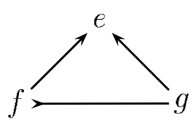
(B2) $e \longrightarrow f \quad \Rightarrow$

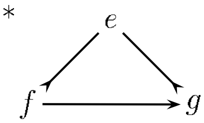


(B2)* $e \succ\!\!\!-\!\!- f \quad \Rightarrow$



The next two axioms tell us that the left [right] arrow is preserved under multiplication, when defined, on the right [left].

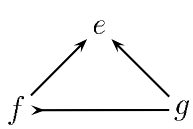
(B3)  $\Rightarrow$ $fe \succ\!\!\!-\!\!\!- ge$

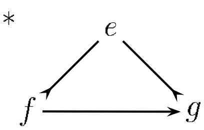
(B3)*  $\Rightarrow$ $ef \longrightarrow eg$

The next four axioms are special cases of associativity of multiplication. In (B4) observe that (B1) is applied implicitly to guarantee that the products eg and $(eg)f$ are defined. In (B5) the existence of the product $(ge)(fe)$ is guaranteed by (B3) and (B1).

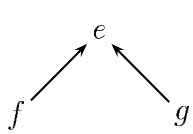
(B4) $e \longrightarrow f \longrightarrow g \quad \Rightarrow \quad (eg)f = ef$

(B4)* $e \succ\!\!\!-\!\!\!- f \succ\!\!\!-\!\!\!- g \quad \Rightarrow \quad f(ge) = fe$

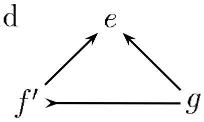
(B5)  $\Rightarrow$ $(gf)e = (ge)(fe)$

(B5)*  $\Rightarrow$ $e(fg) = (ef)(eg)$

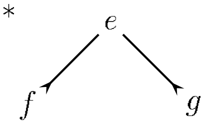
The last two axioms are existential and provide an analogue of Green's Lemma for idempotents. Green's Lemma is discussed and proved in the next section.

(B6)  and $fe \succ\!\!\!-\!\!\!- ge$

$\Rightarrow \exists f' \in E$ such that $f'e = fe$ and

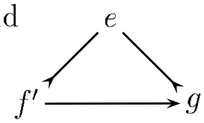


(B6)*



and $ef \longrightarrow eg$

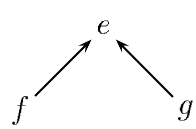
$\Rightarrow \exists f' \in E$ such that $ef' = ef$ and



In Nambooripad's original formulation of the axioms in [24, 25], in place of (B6) and (B6)* above, axioms are given in terms of sandwich sets, which are defined in Section 7.1. The formulation given here was preferred by Easdown [8], and is equivalent to the original by [25, Proposition 2.4]. In [25, Section 2], Nambooripad claims that examples can be given to show all axioms are irredundant.

4.1 Biordered sets of semigroups

Let S be a semigroup and put $E = E(S)$. Then defining $\longrightarrow$ and $\succ\!\!-\!\!-$ as in Example 3.2, it is easy to check from the associativity of S that the dually ordered set $(E, \rightarrow, \succ\!\!-)$ is in fact biordered. This is perhaps least obvious in the case of (B6) and (B6)*, the first of which we will now verify. Assume there exist $e, f, g \in E(S)$ such that



and $fe \succ\!\!-\!\!- ge$.

Consider the product fg in S. We have $fgfg = fegefg = fefg = ffg = fg$, showing that $fg \in E(S)$. Also, $fge = fege = fe$, and clearly $e \longleftarrow fg \succ\!\!-\!\!- g$. So if we put $f' = fg$, we have exhibited an element of $E(S)$ with the properties required by (B6).

We call $E(S)$ the *biordered set of the semigroup* S. Axiom (B1) ensures that we have a partial multiplication inherited from S, with the domain given by all pairs of idempotents such that one is a left or a right zero for the other. In the case that S is a group, the information embodied in the biordered set $E(S)$ is trivial. It is interesting nevertheless that for large classes of semigroups S which are not groups, much (if not all) of the structure of S can be recovered from its biordered set.

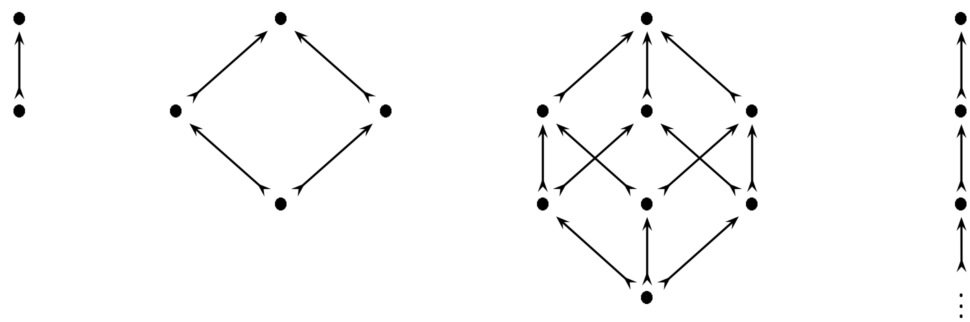
Easdown [12] proved that abstract biordered sets, as defined by the above axioms, are precisely biordered sets of semigroups up to isomorphism. The proof relies on an elaborate word argument involving a semigroup presentation, and in fact produces the freest semigroup with idempotents forming a given biordered set E. This is defined in [12], and studied by Pastijn in [26] under the assumption, redundant in light of [12], that E is isomorphic to the biordered set of some semigroup. These results provide an answer to the question posed at the end of Chapter 3, and lead us to consider what properties of a semigroup (if any) can be determined by investigating its biordered set.

4.2 Illustrations

Example 4.1. Let $(E, \leq)$ be a partially ordered set. We noted in Section 3 that E is dually ordered where $\longrightarrow \;=\; \succ\!\!\!\relbar\!\!\relbar \;=\; \leq$. Now define $D_E = \{(e,f) \mid e \leq f \text{ or } f \leq e\}$ and, for $(e,f) \in D_E$,

$$ef \;=\; \begin{cases} e & \text{if } e \leq f \\ f & \text{if } f \leq e. \end{cases}$$

It is routine to verify the axioms so that $(E, \leq)$ is a biordered set. In particular, all semilattices are biordered sets, and the Hasse diagrams of Examples 2.3 and 2.4 may be depicted as follows:



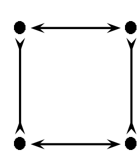
By Proposition 2.1, any semilattice arises as a semigroup of commuting idempotents, so the verification of the biordered set axioms is also immediate by Section 4.1. In fact, any partially ordered set $(E, \leq)$ arises (up to isomorphism) as the biordered set of the semigroup given by the semigroup presentation

$$\langle E \mid xy \ = \ x \ = \ yx \ \text{ wherever } x \leq y \rangle .$$

This is not hard to verify directly, and is a special case of the work of Easdown [12], referred to at the end of the previous subsection. $\square$

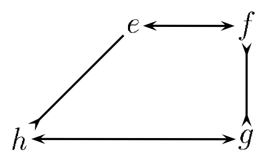
Example 4.2. If E is a left [right] zero semigroup (defined in Section 1.1), then E is its own biordered set where $\succ\!\!-\!\!-$ [$\longrightarrow$] is the universal relation and $\longrightarrow$ [$\succ\!\!-\!\!-$] is the identity relation. For example, the two element left and right zero semigroups have biordered sets which may be depicted by

respectively. The direct product of a left zero with a right zero semigroup is a rectangular band (as observed in Section 1.1), whose biordered set in the two-by-two case may be depicted thus:

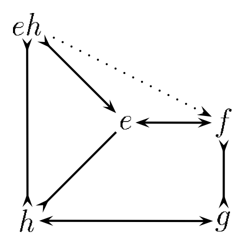


Note that in a rectangular band the product of diagonally opposite elements is missing in the data of the biordered set, because there is no arrow between them. (Nevertheless, these missing products may be recovered by applying the ρ, λ mappings defined below in Section 8.) □

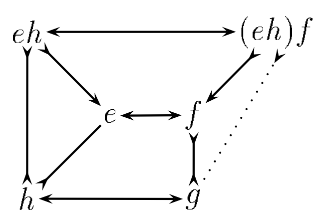
Example 4.3. Consider the following diagram of arrows, obtained by deleting one left arrow from the rectangle in the previous example:



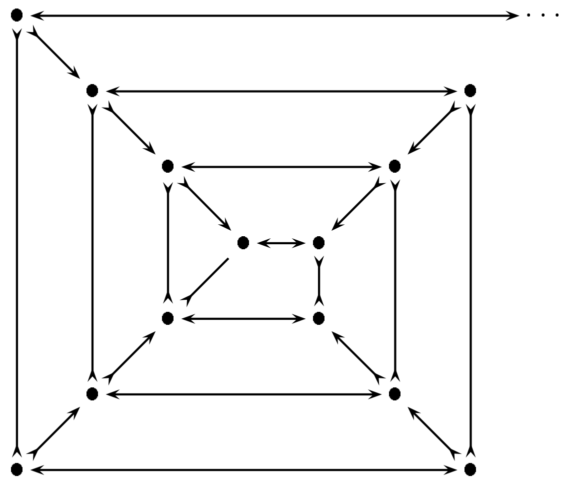
If this were to be part of a biordered set, then the product eh would have to be defined and, to satisfy axiom (B3)*, the following arrows would exist:



By transitivity of $\longrightarrow$, we would have $ef \longrightarrow f$, so the product $(eh)f$ would be defined. To satisfy axiom (B3), the following arrows would exist:



Continuing in this way, we would obtain an infinite spiral:

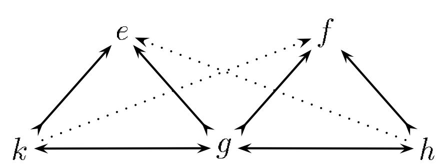


Now, given this diagram, and allowing for the transitive closure of the arrows, it is straightforward to define products in such a way that the biordered set axioms are satisfied. Indeed, the choice of products, as motivated by the way we generated the spiral, is uniquely determined by axioms (B3) and (B3)*. We obtain the *four spiral biordered set* studied by Byleen, Meakin and Pastijn [3]. In fact, this arises as the biordered set of the *four spiral semigroup* defined by the presentation

$$\langle x, y, z, w \mid x^2 = x,\ y^2 = y,\ z^2 = z,\ w^2 = w,$$
$$xy = y,\ yx = x,\ yz = y,\ zy = z,\ wz = z,\ zw = w,\ wx = w\rangle,$$

studied from several points of view in [3]. We will perform some calculations in this semigroup later after introducing the ρ, λ mappings. □

Example 4.4. The diagram of arrows between elements of a biordered set need not uniquely determine the biordered set products. The following diagram (presented first in [13]) defines a dually ordered set



In order for this to become a biordered set we would need to define products arising from the presence of arrows. All of these products are clear from the diagram except for kf and he. There are four choices leading to the satisfiability of the biordered set axioms:

$$E_1: \quad kf = g = he$$

$$E_2: \quad kf = h, \ he = k$$

$$E_3: \quad kf = g, \ he = k$$

$$E_4: \quad kf = h, \ he = g$$

We will see later that E_1 arises as the biordered set of a band. We will also see that E_3, which is clearly isomorphic to E_4, arises as the biordered set of a finite semigroup which cannot be regular, and E_2 is not the biordered set of any finite semigroup. □

5 Green's Relations

Green's relations are a set of equivalence relations defined on semigroups. They reduce to the universal relation on groups, which is why they are not studied in group theory. Green's relations give important insights into the structure of semigroups in general, and provide a framework for investigating the central role played by idempotents in determining this structure. They allow us to construct 'eggbox' diagrams (Section 5.3), which provide a means for visualising relationships between elements of a semigroup. They also lead to Green's Lemma, as well as several other important results needed in this essay. The concept of transitive closure and some initial observations are needed before defining Green's relations.

5.1 Observations

Observe that the classes of reflexive, symmetric and transitive relations are each closed under arbitrary intersections. It follows immediately that if ρ is a relation, then there is a smallest transitive relation ρ^∞ (called the *transitive closure* of ρ), and also a smallest equivalence $\overline{\rho}$ containing ρ. Clearly,

$$\overline{\rho^\infty} = \overline{\rho} \qquad \text{and} \qquad \rho^\infty = \bigcup_{n=1}^{\infty} \rho^n .$$

Lemma 5.1. If λ and ρ are commuting equivalence relations (that is, $\lambda \circ \rho = \rho \circ \lambda$), then $(\lambda \circ \rho)^\infty = (\lambda \circ \rho)$.

Proof. Clearly, by commutativity and transitivity of the equivalence relations, $(\lambda \circ \rho)^n = \lambda^n \circ \rho^n = \lambda \circ \rho$ for each n, so that

$$(\lambda \circ \rho)^\infty = \bigcup_n (\lambda \circ \rho)^n = \bigcup_n \lambda \circ \rho = \lambda \circ \rho.$$

□

5.2 Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$

We write $a\mathcal{L}b$ if there exist $x, y \in S^1$ such that $xa = b$ and $yb = a$, equivalent to $S^1a = S^1b$. We write $a\mathcal{R}b$ if there exist $u, v \in S^1$ such that $au = b$ and $bv = a$, equivalent to $aS^1 = bS^1$. Green's relations $\mathcal{L}$ and $\mathcal{R}$ express the idea of right and left mutual divisibility, respectively. Notice that $\mathcal{L}$ on the dual S^* corresponds to $\mathcal{R}$ on S.

We define Green's relation $\mathcal{J}$ as the two-sided analogue of $\mathcal{L}$ and $\mathcal{R}$, that is, $a\mathcal{J}b$ if there exist $x, y, u, v \in S^1$ such that $b = xay$ and $a = ubv$, equivalent to $S^1xS^1 = S^1yS^1$. We have $a\mathcal{J}b$ whenever we can get from a to b and b to a by multiplying, possibly simultaneously, on the left and right. The two remaining Green's relations build on the definitions of $\mathcal{L}$ and $\mathcal{R}$. We let $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \overline{\mathcal{L} \cup \mathcal{R}}$ (the smallest equivalence containing both $\mathcal{L}$ and $\mathcal{R}$).

From the definitions, it is easy to see that $\mathcal{L}$ and $\mathcal{R}$ are equivalences on S, and thus $\mathcal{H}$ is an equivalence. It is also easy to see that $\mathcal{J}$ is an equivalence, and we have that $\mathcal{D}$ is an equivalence by definition. Henceforth, when x is an element of some semigroup, we take L_x, R_x, H_x, D_x, and J_x to refer to the $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$-class containing x, respectively. To be able to work with $\mathcal{D}$, we would like to find an explicit expression for it. To do this, we need the following fact:

Proposition 5.2. For any semigroup S, $\mathcal{L}$ and $\mathcal{R}$ commute, that is, $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

Proof. Take any $(a, b) \in \mathcal{L} \circ \mathcal{R}$. Then $a\mathcal{L}c$ and $c\mathcal{R}b$ for some $c \in S$. Hence

$$xa = c,\ yc = a\ (\exists x, y \in S^1) \quad \text{and} \quad cu = b,\ bv = c\ (\exists u, v \in S^1).$$

Observe that $au = ycu = yb$. Also,

$$a = yc = ybv = (ycu)v \qquad \text{and} \qquad b = cu = xau = x(ycu),$$

so $a\mathcal{R}(ycu)\mathcal{L}b$, that is, $(a, b) \in \mathcal{R} \circ \mathcal{L}$. Hence $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$. By duality, we get $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$, whence equality. □

We now have all the results needed to identify $\mathcal{D}$ explicitly.

Proposition 5.3. For any semigroup S, $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

Proof. Observe that $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{D} \circ \mathcal{D} = \mathcal{D}$, since $\mathcal{L}, \mathcal{R} \subseteq \mathcal{D}$. Now we show that $\mathcal{L} \circ \mathcal{R}$ is an equivalence relation, and that it contains both $\mathcal{L}$ and $\mathcal{R}$. Symmetry follows since $(\mathcal{L} \circ \mathcal{R})^{-1} = \mathcal{R}^{-1} \circ \mathcal{L}^{-1} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. Reflexivity follows since for all $c \in S$, $c\mathcal{L}c\mathcal{R}c$. Since $\mathcal{L}$ and $\mathcal{R}$ commute, we have $\mathcal{L} \circ \mathcal{R} = (\mathcal{L} \circ \mathcal{R})^{\infty}$ by Lemma 5.1, proving transitivity. Thus $\mathcal{L} \circ \mathcal{R}$ is an equivalence relation.

Also, for all $(a, b) \in \mathcal{L}$, $a\mathcal{L}b\mathcal{R}b$, giving $a(\mathcal{L} \circ \mathcal{R})b$, whence $\mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$. By duality, we have $\mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R}$. But $\mathcal{D} = \overline{\mathcal{L} \cup \mathcal{R}}$, so $\mathcal{D} \subseteq \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, whence equality. □

The $\mathcal{J}$-classes of a semigroup S can be partially ordered by the rule $J_x \leq J_y$ if $x \in S^1 y S^1$. This partial order applies to the $\mathcal{D}$-classes in the case that S is finite, as shown by the following proposition.

Proposition 5.4. If S is finite then $\mathcal{J} = \mathcal{D}$, so in particular, the $\mathcal{D}$-classes form a partially ordered set.

Proof. From the definition of $\mathcal{J}$, we have that $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$. Since $\mathcal{D}$ is the smallest equivalence containing $\mathcal{L}$ and $\mathcal{R}$, we must have $\mathcal{D} \subseteq \mathcal{J}$.

Now suppose S is finite and take any $a, b \in S$ such that $a\mathcal{J}b$. This gives us $xay = b$ and $ubv = a$ for some $x, y, u, v \in S^1$. We aim to prove $a\mathcal{D}b$, so we must find some $c \in S^1$ such that $a\mathcal{L}c\mathcal{R}b$. First observe that $a = ubv = u(xay)v$. By repeatedly applying this identity, we obtain $a = (ux)a(yv) = (ux)^2 a(yv)^2 = \cdots = (ux)^m a(yv)^m$, for any value of $m \geq 1$. Because S is finite, we can choose m such that $(ux)^m$ is idempotent. Now let $c = xa$, giving

$$\begin{aligned} a &= (ux)^m a(yv)^m = (ux)^m (ux)^m a(yv)^m \\ &= (ux)^m a = (ux)^{m-1} uxa = (ux)^{m-1} uc. \end{aligned}$$

This proves $a\mathcal{L}c$. By a similar argument to the one above, we can obtain $b = (xu)b(vy) = (xu)^2 b(vy)^2 = \cdots = (xu)^n a(vy)^n$, for any value of $n \geq 1$. We choose n such that $(vy)^n$ is idempotent, giving

$$\begin{aligned} c &= xa = x(ux)^{n+1} a(yv)^{n+1} = (xu)^{n+1} xay(vy)^n v \\ &= (xu)^{n+1} b(vy)^{2n} v = (xu)^{n+1} b(vy)^{n+1}(vy)^{n-1} v = b(vy)^{n-1} v. \end{aligned}$$

Also, $cy = b$, so $c\mathcal{R}b$, giving us $a\mathcal{L}c\mathcal{R}b$. Thus $\mathcal{J} \subseteq \mathcal{D}$, whence equality. $\square$

5.3 Eggboxes

The previous two results suggest a useful diagrammatic representation of the structure of a finite semigroup. For each $\mathcal{D}$-class of a semigroup S, we have

$$x\mathcal{D}y \Rightarrow x\mathcal{L}a\mathcal{R}y \text{ and } x\mathcal{R}b\mathcal{L}y \qquad (\exists a, b \in S).$$

x	$\cdots$	b
$\vdots$		
a		y

This fact allows us to construct a rectangular array (or *eggbox*) representing the $\mathcal{D}$-class, where the columns are $\mathcal{L}$-classes and the rows are $\mathcal{R}$-classes. Each cell of the eggbox is the intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class, which is of course an $\mathcal{H}$-class. As a consequence of Green's lemma (proved in Section 5.4 below), it turns out that the $\mathcal{H}$-classes in a given $\mathcal{D}$-class all have the same cardinality. The existence of the elements a and b guarantees that none of these cells will be empty, so we know that the diagram is an accurate depiction of the set of all $\mathcal{H}$-classes contained in the $\mathcal{D}$-class it represents. Because the $\mathcal{D}$-classes are partially ordered we can depict S as a Hasse diagram in which each node becomes an eggbox. This idea is illustrated by Examples 5.7 and 5.8 below.

Proposition 5.5. Let S be a semigroup, and let H be an $\mathcal{H}$-class of S. Then H contains an idempotent if and only if it is a group.

Proof. If H is a group, then it must contain an identity element, which is an idempotent. Now assume e is an idempotent in H. Let x be an arbitrary element of H. We have $x = ey$ for some y in S^1, so $ex = eey = ey = x$. Similarly $xe = x$. Thus e is an identity for H. We also have some $b, c \in S^1$ such that $bx = e = xc$ since $x\mathcal{H}e$. Hence $(ebe)x = ebx = ee = e$ and

$x(ebe) = xebxc = x(bx)c = xec = xc = e$. Thus $ebe\mathcal{H}e$, so ebe is a two-sided inverse of x lying in H. Take any $f, g \in H$, with inverses $f', g' \in H$ respectively. Then $(fg)(g'f') = fef' = ff' = e$ and similarly $(g'f')(fg) = e$, from which it follows quickly that $fg\mathcal{L}e\mathcal{R}fg$, so $fg\mathcal{H}e$. Hence $\mathcal{H}$ is closed under multiplication, the associativity of which is inherited from S. This proves H is a group. □

Proposition 5.6. A subgroup G of a semigroup S is maximal if and only if $G = H_e$ for some idempotent $e \in S$.

Proof. Say G is a subgroup of a semigroup S, with identity element e. Assume first that G is maximal. For any $g \in G$, $eg = ge = g$ and $gg^{-1} = g^{-1}g = e$, showing that $g \in H_e$. So we have $G \subseteq H_e$, and we know that H_e is a group from Proposition 5.5, so $G = H_e$ since G is maximal.

Now assume $G = H_e$, where e is some idempotent in S. Say there is some subgroup H of S such that $G \subseteq H$, and h is any element of H. The identity element of H must be e, since e is idempotent, $e \in H$, and H is a group. Then $eh = he = h$ and $hh^{-1} = h^{-1}h = e$, which shows $h \in H_e = G$, whence $H \subseteq G$. Thus $H = G$, proving maximality. □

The following two examples involve mappings. If X is a set and $\alpha : X \to X$ is a mapping, we call the size of the image of α its *rank*, denoted by rank α, and call the equivalence relation $\ker \alpha = \{(x, y) \in X \times X \mid x\alpha = y\alpha\}$ the *kernel* of α. Note that the rank of a mapping is also the number of equivalence classes with respect to the kernel.

Example 5.7. The *full transformation semigroup* $\mathcal{T}(X)$ is the set of all functions $X \to X$, for some set X, with the semigroup operation given by composition of mappings. Consider the case where X is finite.

Claim. The $\mathcal{D}$-classes of $\mathcal{T}(X)$ correspond to the sets of functions with equal rank.

Proof. To see this, we first show that $\alpha\mathcal{L}\beta$ if and only if im $\alpha =$ im β, and that $\alpha\mathcal{R}\beta$ if and only if ker $\alpha =$ ker β.

Assume $\alpha\mathcal{L}\beta$ for some $\alpha, \beta \in \mathcal{T}(X)$. Then $\beta = \phi\alpha$ for some $\phi \in \mathcal{T}(X)$. If $x \in$ im α then $x = y\alpha = y\phi\beta$ for some $y \in X$, so $x \in$ im β. Thus im $\alpha \subseteq$ im β. Similarly, im $\beta \subseteq$ im α, whence im $\alpha =$ im β. Conversely, if im $\alpha =$ im β, it is easy to construct functions ϕ and ψ to give us $\phi\alpha = \beta$ and $\psi\beta = \alpha$, whence $\alpha\mathcal{L}\beta$.

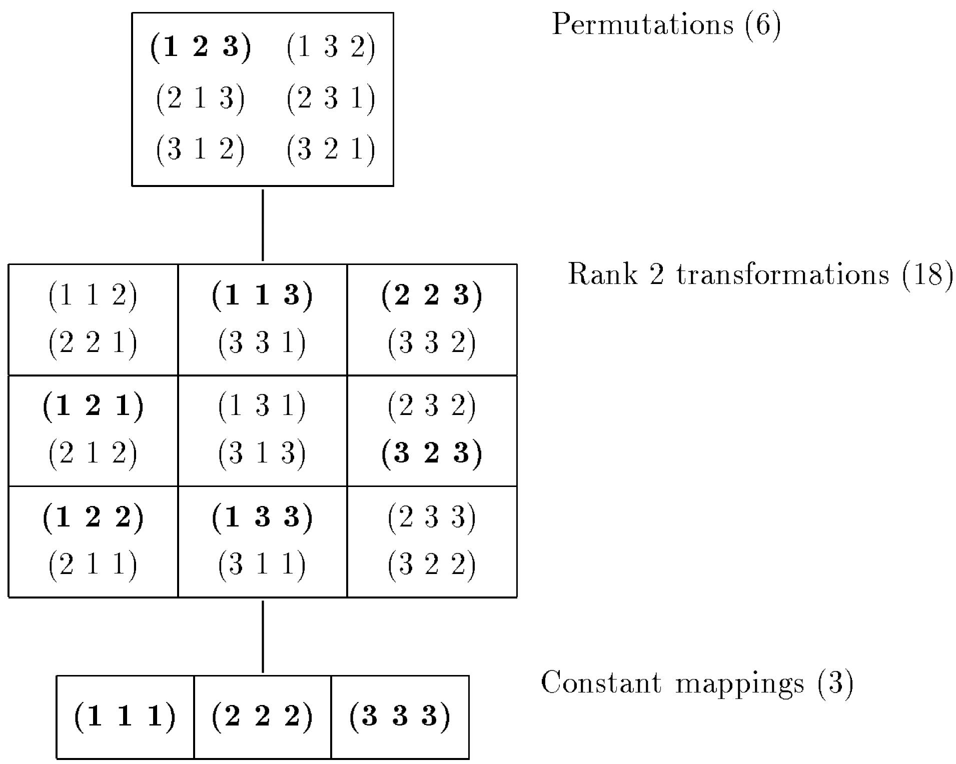
Now assume $\alpha\mathcal{R}\beta$ for some $\alpha, \beta \in \mathcal{T}(X)$. Then $\beta = \alpha\phi$ for some $\phi \in \mathcal{T}(X)$. If $(x, y) \in$ ker α, then $x\alpha = y\alpha$, so $x\beta = x\alpha\phi = y\alpha\phi = y\beta$, giving $x\beta = y\beta$, that is, $(x, y) \in$ ker β. Thus ker $\alpha \subseteq$ ker β. Similarly, we obtain ker $\beta \subseteq$ ker α, whence ker $\alpha =$ ker β. Conversely, if ker $\alpha =$ ker β, we can find functions (which may be permutations) ϕ and ψ such that $\beta = \alpha\phi$, $\alpha = \beta\psi$, whence $\alpha\mathcal{R}\beta$.

Consider two arbitrary functions α and β. If $\alpha\mathcal{D}\beta$, then $\alpha\mathcal{L}\gamma\mathcal{R}\beta$ for some $\gamma \in \mathcal{T}(X)$, so that im $\alpha =$ im γ and ker $\gamma =$ ker β, and then rank $\alpha =$ rank $\gamma =$ rank β. Conversely, suppose rank $\alpha =$ rank $\beta = r$, for some r. Let im $\alpha = \{k_1, \ldots, k_r\}$, im $\beta = \{l_1, \ldots, l_r\}$. Define

$$\gamma : \quad \begin{cases} k_i \mapsto l_i & \text{for } i = 1, \ldots, r \\ z \mapsto l_1 \text{ (or anything)} & \text{for } z \notin \text{im } \alpha. \end{cases}$$

Then im $\alpha\gamma =$ im β, so $\alpha\gamma\mathcal{L}\beta$. By a similar argument, we can find δ such that $\beta\delta\mathcal{L}\alpha$. Hence $J_\alpha \leq J_\beta \leq J_\alpha$, so $J_\alpha = J_\beta$, whence $D_\alpha = D_\beta$, by Proposition 5.4. □

The eggbox diagram of $\mathcal{T}(X)$ for the case $X = [3]$ is illustrated below, with each $\mathcal{D}$-class labelled, and its size given in parentheses. For each mapping $\alpha \in \mathcal{T}(X)$, it is typical to write $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1\alpha & 2\alpha & 3\alpha \end{pmatrix}$, and further abbreviate this to $(1\alpha\ 2\alpha\ 3\alpha)$. The reader should be careful not to confuse this with the cycle notation used for permutations.



Claim. For any finite X, the $\mathcal{D}$-classes of $\mathcal{T}(X)$ form a chain.

Proof. We know the $\mathcal{D}$-classes are partially ordered by the rule $D_\alpha \leq D_\beta$ if and only if $\alpha \in \mathcal{T}(X)\beta\mathcal{T}(X)$. We have also seen that for any $\alpha, \beta \in \mathcal{T}(X)$, $D_\alpha = D_\beta$ if and only if rank α = rank β. If rank $\alpha <$ rank β, then by collapsing elements in the image, we can find a mapping γ such that rank $\beta\gamma$ = rank α, and then we have $D_\alpha = D_{\beta\gamma} < D_\beta$. Hence all $\mathcal{D}$-classes are comparable, thus forming a chain. □

Example 5.8. Let X be a set. Consider the symmetric inverse semigroup $\mathcal{I}(X)$ defined in Section 2.1. That $\mathcal{I}(X)$ is an inverse semigroup is proved in Chapter 6, together with the fact that inversion (in this case coinciding with the usual inversion of partial one-one mappings) is an anti-isomorphism. As with $\mathcal{T}(X)$, the $\mathcal{D}$-classes of $\mathcal{I}(X)$ are the sets of partial one-one mappings with equal rank. This follows by an argument similar to that for $\mathcal{I}(X)$, noting that $\alpha\mathcal{L}\beta$ if and only if im α = im β, and $\alpha\mathcal{R}\beta$ if and only if $\alpha^{-1}\mathcal{L}\beta^{-1}$, equivalent to dom α = im α^{-1} = im β^{-1} = dom β.

The eggbox diagram of $\mathcal{I}(X)$ for the case $X = [3]$ is illustrated below. For each mapping $\alpha \in \mathcal{I}(X)$, we represent α by the sequence $(1\alpha\ 2\alpha\ 3\alpha)$, putting a dash where the mapping is undefined. Idempotents are indicated by the use of boldface. Observe that the idempotents are precisely the restrictions of the identity mapping to subsets of X.

Permutations (6)

(1 2 3)	(1 3 2)
(2 1 3)	(2 3 1)
(3 1 2)	(3 2 1)

Rank 2 mappings (18)

(– 1 2) (– 2 1)	(– 1 3) (– 3 1)	**(– 2 3)** (– 3 2)
(1 – 2) (2 – 1)	**(1 – 3)** (3 – 1)	(2 – 3) (3 – 2)
(1 2 –) (2 1 –)	(1 3 –) (3 1 –)	(2 3 –) (3 2 –)

Rank 1 mappings (9)

(– – 1)	(– – 2)	**(– – 3)**
(– 1 –)	**(– 2 –)**	(– 3 –)
(1 – –)	(2 – –)	(3 – –)

Zero mapping (1)

(– – –)

5.4 Green's Lemma

Before introducing Green's Lemma, we need to define right and left translations. For an element x of a semigroup S, the *right [left] translation* ρ_x $[\lambda_x]$ is the map $S \to S$ given by $s\rho_x = sx$ $[s\lambda_x = xs]$, for all s in S. We will also refer to any restriction of ρ_x or λ_x to a subset of S as a *right* or *left translation* respectively.

Lemma 5.9. (Green's Lemma, $\mathcal{R}$-version) Let S be a semigroup with $a, b \in S$ and $a\mathcal{R}b$, giving us $b = as$, $a = bs'$ for some $s, s' \in S^1$. Then the right translations $\rho_s|_{L_a}$ and $\rho_{s'}|_{L_b}$ are mutually inverse $\mathcal{R}$-class preserving bijections from L_a onto L_b, and from L_b onto L_a respectively.

Proof. The translation $\rho_s : S \to S$ takes a to b, since $as = b$. Let x be an arbitrary element of L_a. Then $x = ya$ for some $y \in S^1$ and $a = zx$ for some $z \in S^1$. So $x\rho_s = xs = y(as)$ and $as = z(xs)$, showing that $x\rho_s\mathcal{L}as$. This gives us $x\rho_s \in L_{as} = L_b$, showing that ρ_s maps L_a into L_b. Similarly, $\rho_{s'}$ maps L_b into L_a. Now consider the following composition applied to x:

$$x(\rho_s \circ \rho_{s'}) \;=\; ya(\rho_s \circ \rho_{s'}) \;=\; yass' \;=\; ybs' \;=\; ya \;=\; x.$$

This shows that $(\rho_s \circ \rho_{s'})|_{L_a} = \mathrm{id}_{L_a}$. Similarly, $(\rho_{s'} \circ \rho_s)|_{L_b} = \mathrm{id}_{L_b}$, proving that $\rho_s|_{L_a}$ and $\rho_{s'}|_{L_b}$ are mutually inverse bijections. Further, $x\rho_s = xs$ and $x = x(\rho_s \circ \rho_{s'}) = x\rho_s s'$, which gives $x\rho_s\mathcal{R}x$, showing that $\rho_s|_{L_a}$ is $\mathcal{R}$-class preserving. Similarly, we can see that $\rho_{s'}|_{L_b}$ also preserves $\mathcal{R}$-classes. $\square$

Lemma 5.10. (Green's Lemma, $\mathcal{L}$-version) Let S be a semigroup with $a, b \in S$ and $a\mathcal{L}b$, giving us $b = ta$, $a = t'b$ for some $t, t' \in S^1$. Then the left translations $\lambda_t|_{R_a}$ and $\lambda_{t'}|_{R_b}$ are mutually inverse $\mathcal{L}$-class preserving bijections from R_a onto R_b, and from R_b onto R_a respectively.

Proof. This result follows by duality from the previous lemma. $\square$

Applying Green's Lemma to the eggbox diagram, we see that multiplication on the right gives us right translations, which are invertible mappings taking one column to another, while preserving rows. Similarly, left multiplication produces left translations which translate rows without disturbing the columns. This lemma is crucial to the proof of Theorem 7.5 later.

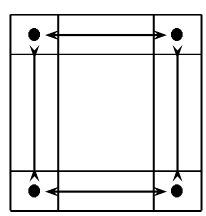
Corollary 5.11. The $\mathcal{H}$-classes in any $\mathcal{D}$-class of a semigroup all have the same cardinality.

Proof. Let $x, y \in S$ for some semigroup S. Assume $D_x = D_y$. Then there exists some z such that $x\mathcal{L}z\mathcal{R}y$. Using Lemma 5.10, there is an $\mathcal{L}$-class preserving bijection λ from R_x to R_z, and by Lemma 5.9 there is an $\mathcal{R}$-class preserving bijection ρ from L_z to L_y. The composition (as partial maps) $\lambda \circ \rho$ is a bijection between H_x and H_y, so these $\mathcal{H}$-classes must be the same size. □

5.5 Green's relations restricted to idempotents

It is now useful to consider the connection between eggboxes, Green's relations $\mathcal{L}$ and $\mathcal{R}$, and the arrow relations $\succ\!\!-$ and $\longrightarrow$ defined earlier. Let e and f be idempotents of a semigroup S. If $e\mathcal{L}f$, then we have $e = xf$ for some $x \in S$, so $ef = xff = xf = e$, and similarly $f = fe$, which is equivalent to writing $e \succ\!\!-\!\!\prec f$. Conversely $\succ\!\!-\!\!\prec$ implies $e = ef$ and $f = fe$ so that $e\mathcal{L}f$. By duality, $e\mathcal{R}f$ is equivalent to $e \longleftrightarrow f$. Thus the arrow pre-orders $\succ\!\!-$ and $\longrightarrow$ on sets of idempotents can be thought of as 'bisecting' Green's relations $\mathcal{L}$ and $\mathcal{R}$ respectively. More precisely, $\succ\!\!-\!\!\prec \; = \mathcal{L}|_{E\times E}$ and $\longleftrightarrow \; = \mathcal{R}|_{E\times E}$, where $E = E(S)$.

We can use arrows and eggboxes in the same diagram. For example, in a rectangular band we have relationships between idempotents arranged in 'diagonally opposite' pairs as depicted in the following diagram.



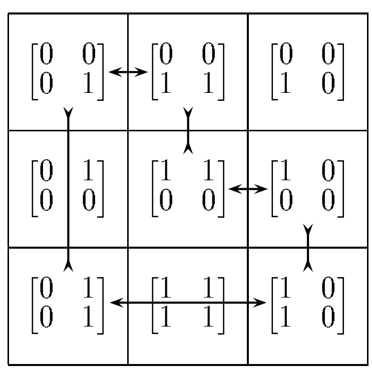
From what we have just shown, in any eggbox diagram there are implicit double left arrows between all idempotents lying in the same column, and double right arrows between those lying in the same row.

5.6 Biordered sets as the skeleton of a semigroup

If all the elements of a semigroup S are arranged into eggboxes, the idempotents and arrows forming the biordered set of S can be thought of as a skeleton of the semigroup as a whole — a remnant structure that survives after all non-idempotent elements have been removed. A clear illustration of this is the semigroup of 2×2 matrices over the field $\mathbb{Z}_2$, examined below. In this example, the group of units of the semigroup turns out to be the group of symmetries of the biordered set.

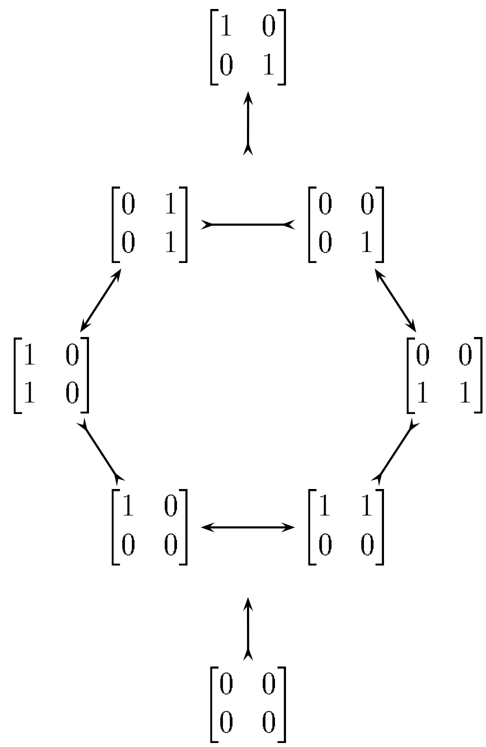
Example 5.12. The set $\mathrm{Mat}_2(\mathbb{Z}_2)$ of all 2×2 matrices over the field $\mathbb{Z}_2$ forms a semigroup of 16 elements under matrix multiplication. There are 8 idempotents: the identity matrix, the zero matrix, and six rank 1 matrices. The zero matrix sits in its own eggbox at the bottom of the order of $\mathcal{D}$-classes since it is both a left and right zero for all elements of the semigroup. The identity matrix is at the top, and its $\mathcal{D}$-class is the group of units. All elements obviously act as left and right zeros to the identity element of a monoid, as in this case.

The only $\mathcal{D}$-class containing more than one idempotent is that of the rank 1 matrices. We reproduce its eggbox diagram below, with the arrow relations between all idempotents of this $\mathcal{D}$-class depicted. The relative positions of the elements are determined by the fact that in $\mathrm{Mat}_2(\mathbb{Z}_2)$, two matrices lie in the same $\mathcal{R}$-class [$\mathcal{L}$-class] if and only if they have the same column [row] space.



The symmetries of this skeleton become more obvious when the diagram is rearranged. For completeness, the identity and zero matrices are also included.

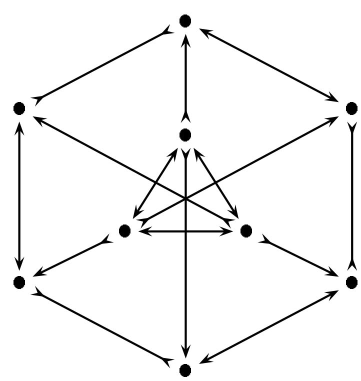
Note that the arrows attached to these two matrices each represent a set of six separate arrows connected to all six matrices of the central hexagon. It is clear that this abbreviation does not alter the symmetry properties of the skeleton.



With six sides alternating between double left and right arrows, this biordered set has the symmetries of a triangle. The group describing these symmetries is Sym(3), which is isomorphic to $GL_2(\mathbb{Z}_2)$, the group of units of our semigroup $Mat_2(\mathbb{Z}_2)$. It was observed in Section 1.1 that conjugating idempotents by an element of the group of units produces another idempotent. Here elements of $GL_2(\mathbb{Z}_2)$ permute the rank 1 idempotents under conjugation. Now $GL_2(\mathbb{Z}_2)$ is generated by $\alpha = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Observe that conjugating by α reflects the diagram in the vertical axis, and conjugation by β rotates the diagram by 120 degrees. It follows quickly that the conjugation action of $GL_2(\mathbb{Z}_2)$ on the biordered set is faithful.

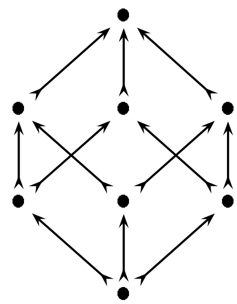
Here we have been looking only at the operation of matrix multiplication. If we also consider addition, then $Mat_2(\mathbb{Z}_2)$ is a ring. The question of characterising idempotents of the multiplicative semigroups of rings (posed and explored in [14], and for regular rings in [27]) is open, and the conjugation action of the group of units on the underlying biordered set has not, to the author's knowledge, been systematically investigated. □

Example 5.13. The eggbox diagram for $\mathcal{T}([3])$ is given in Example 5.7, with idempotents highlighted. By considering products, it is straightforward to extract the following arrow diagram, where the idempotents (ignoring the identity) have been contracted to points, and the constant mappings are displayed in a central triangle.



Again we have the symmetries of the triangle. The group of units of $\mathcal{T}([3])$ is Sym(3), and as in the previous example, we get a faithful representation of Sym(3) by conjugation action on the hexagon using rotations and reflections. We also get a faithful representation of Sym(3) by conjugation action on the $\mathcal{R}$-class of constant mappings displayed in the central triangle, which amounts to reproducing permutations of a 3 element set. □

Example 5.14. The eggbox diagram for $\mathcal{I}([3])$ is given in Example 5.8, with idempotents highlighted. We obtain the following skeleton, an instance of Example 4.1, where the idempotents are again represented by points.



As in the two previous examples, the group of units of $\mathcal{I}([3])$ is Sym(3), and this time the conjugation action of elements of the group of units is to permute the atoms (the third layer) and also the co-atoms (the second layer). $\square$

6 Idempotents of Inverse Semigroups

Containing only one idempotent, the biordered set of a group is trivial. The most widely studied class of semigroups (after groups) is that of inverse semigroups, introduced independently but almost simultaneously by Vagner and Preston [29, 32]. They are also the class of semigroups most similar to groups, and for this reason were originally referred to as 'generalised groups' [32]. Just as abstract groups arise as groups of permutations, inverse semigroups may be faithfully represented by semigroups of partial one-one mappings of a set closed under inversion, by an adaptation of Cayley's Theorem known as the Vagner-Preston Theorem [20, Theorem 1.10].

By Proposition 2.1 above and Proposition 6.4 below, any semilattice may be regarded as an inverse semigroup of commuting idempotents. By Corollary 6.2 below, the biordered set of an inverse semigroup is a semilattice. Thus an abstract biordered set arises as the biordered set of an inverse semigroup if and only if it is a semilattice.

Proposition 6.1. The idempotents of an inverse semigroup commute.

Proof. Following Howie [20], let S be an inverse semigroup, and take any $e, f \in E(S)$. Notice that $f(ef)^{-1}e$ is idempotent, since

$$(f(ef)^{-1}e)^2 \ = \ f[(ef)^{-1}ef(ef)^{-1}]e \ = \ f(ef)^{-1}e.$$

Also, ef is an inverse of $f(ef)^{-1}e$ since

$$[f(ef)^{-1}e]ef[f(ef)^{-1}e] \ = \ f(ef)^{-1}ef(ef)^{-1}e \ = \ f(ef)^{-1}e$$

and

$$ef[f(ef)^{-1}e]ef \ = \ ef(ef)^{-1}ef \ = \ ef.$$

An idempotent is its own unique inverse in an inverse semigroup, so we must have $f(ef)^{-1}e = ef$, showing that ef is also idempotent. By a similar argument, we see that fe is idempotent, and we obtain

$$(ef)(fe)(ef) \ = \ (ef)^2 \ = \ ef \qquad \text{and} \qquad (fe)(ef)(fe) \ = \ (fe)^2 \ = \ fe.$$

Thus ef is an inverse of fe, and both elements are idempotent. Hence $ef = fe$. □

Corollary 6.2. The set of idempotents of an inverse semigroup form a lower semilattice.

Proof. Since idempotents commute it is easy to see that the product of two idempotents in an inverse semigroup S is another idempotent, so the set of idempotents of S forms a commutative semigroup. The result follows by Proposition 2.2. $\Box$

Corollary 6.3. Inversion is an anti-isomorphism of an inverse semigroup.

Proof. Let $x, y \in S$ for some inverse semigroup S. Then $x^{-1}x$ and $y^{-1}y$ are idempotents, so commute by Proposition 6.1. The following two calculations show that $(xy)^{-1} = y^{-1}x^{-1}$.

$$xy(y^{-1}x^{-1})xy \;=\; x(yy^{-1})(x^{-1}x)y \;=\; x(x^{-1}x)(yy^{-1})y \;=\; xy$$

$$(y^{-1}x^{-1})(xy)(y^{-1}x^{-1}) \;=\; y^{-1}yy^{-1}x^{-1}xx^{-1} \;=\; y^{-1}x^{-1}$$

$\Box$

Proposition 6.4. Any regular semigroup with commuting idempotents is inverse.

Proof. Let S be a regular semigroup with commuting idempotents. Let $s \in S$ with inverses x, y. The inverse identity $sxs = s$ implies that sx is idempotent, since $(sx)^2 = (sxs)x = sx$. Similarly, the identity $sys = s$ implies that sy is idempotent. The same identities give us $sx\mathcal{R}s$ and $s\mathcal{R}sy$, from which we obtain $sx\mathcal{R}sy$, allowing us to write $sx \longleftrightarrow sy$. Thus both idempotents act as right zeros for each other, and since the idempotents of S commute, we have $sx = (sy)(sx) = (sx)(sy) = sy$. Dually $xs \rightarrowtail\!\!\leftarrow ys$, and we obtain $xs = ys$. Using the remaining inverse identities, $x = x(sx) = (xs)y = ysy = y$, showing that the inverse of s is unique. Hence S is an inverse semigroup. $\Box$

Proposition 6.5. If S is a semigroup, then S is inverse if and only if S is regular and the idempotents of S commute.

Proof. This follows directly from Propositions 6.1 and 6.4. □

Example 6.6. Let X be a set. We claim that $\mathcal{T}(X)$ is regular and $\mathcal{I}(X)$ is inverse.

Proof. Let $\alpha \in \mathcal{T}(X)$. Define a mapping $\beta : X \to X$ by

$$y \quad \mapsto \quad \begin{cases} \text{any element of } \{x \in X \mid x\alpha = y\} & \text{if } y \in \operatorname{im} \alpha \\ \text{any element of } X & \text{if } y \notin \operatorname{im} \alpha \end{cases}$$

Then $\beta \in \mathcal{T}(X)$ and it is easy to see that $\alpha\beta\alpha = \alpha$. Thus α is regular (by an observation in Section 1.1), whence $\mathcal{T}(X)$ is regular.

Using a similar construction, we can find an inverse β for any $\alpha \in \mathcal{I}(X)$, so $\mathcal{I}(X)$ is regular. Observing that the idempotents of $\mathcal{I}(X)$ are all restrictions of the identity mapping, and thus commute, we can apply Proposition 6.4 to conclude that $\mathcal{I}(X)$ is inverse. □

Example 6.7. The bicyclic semigroup (introduced as Example 2.4) is regular because $y^j x^i$ is an inverse of $y^i x^j$, for all $i, j \in \mathbb{N}$. The idempotents of this semigroup commute, since $e_i e_j = e_{\max\{i,j\}} = e_j e_i$ for all $i, j \geq 0$, so, again by Proposition 6.4, the bicyclic semigroup is inverse. □

7 Idempotents of Regular Semigroups

The next most widely studied class beyond inverse semigroups is the class of regular semigroups, a further generalisation. We noted in Example 6.6 that the semigroup $\mathcal{T}(X)$ of transformations of a set X is regular. There is a result (due to Lallement [21], and explained in [20]) analogous to the Vagner-Preston Theorem, stating that a regular semigroup S can be faithfully represented by a regular subsemigroup of $\mathcal{T}(S) \times \mathcal{T}^*(S)$.

Biordered sets were invented by Nambooripad [24, 25] in his successful characterisation of the partial algebras of idempotents of regular semigroups. It turned out that he needed to add just one more axiom to those given in Section 4 to achieve this characterisation. To state the extra axiom, we need first to introduce the concept of sandwich sets.

7.1 Sandwich Sets of Pairs

The notion introduced in this subsection is a special case of that in Section 7.2, but is discussed separately because of its historical significance, and because the formulation simplifies considerably. The reason for the use of the word 'sandwich' will become clear after Proposition 7.3.

Let E be a biordered set. We define

$$\mathcal{U}(e,f) = \{g \in E \mid e \relbar\joinrel\prec g \longrightarrow f\}$$

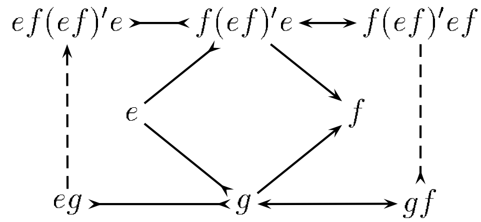
and

$$\mathcal{S}(e,f) = \{h \in \mathcal{U}(e,f) \mid \forall g \in \mathcal{U}(e,f)\ eg \longrightarrow eh \text{ and } gf \succ\joinrel\relbar hf\},$$

which we call the *sandwich set* of e and f. We say that E is *regular* if $\mathcal{S}(e,f)$ is non-empty and $\mathcal{S}(e,f)$ is non-empty for all $e, f \in E$. This is Nambooripad's further axiom, introduced in [24, 25], and the use of the word 'regular' is justified by the following two results.

Proposition 7.1. If S is a regular semigroup, then the biordered set $E(S)$ is regular.

Proof. Suppose S is a regular semigroup and take any $e, f \in E(S)$. Let $(ef)'$ be an inverse of ef. Observe that the element $f(ef)'e$ is idempotent, and that $e \mathrel{-\!\!\!-\!\!\!\prec} f(ef)'e \longrightarrow f$, so $f(ef)'e \in \mathcal{U}(e,f)$. We aim to show that $f(ef)'e \in \mathcal{S}(e,f)$. Let $g \in \mathcal{U}(e,f)$. Then it is easy to see that we have the following relations indicated by the solid arrows.



We obtain the first dashed arrow by observing $[ef(ef)'e]eg = ef(ef)'eg = ef(ef)'efg = efg = eg$, and the second from $gf[f(ef)'ef] = gf(ef)'ef = gef(ef)'ef = gef = gf$. This proves that $f(ef)'e$ is in the sandwich set of e and f. Hence $\mathcal{S}(e,f)$ is non-empty, proving that $E(S)$ is regular. □

Theorem 7.2. If E is a regular biordered set then there exists a regular semigroup S such that $E = E(S)$.

Proof. This theorem is due to [25]. Its proof is difficult and beyond the scope of this essay. An alternative proof using ρ and λ mappings (defined in Section 8) was given by Easdown in [8]. □

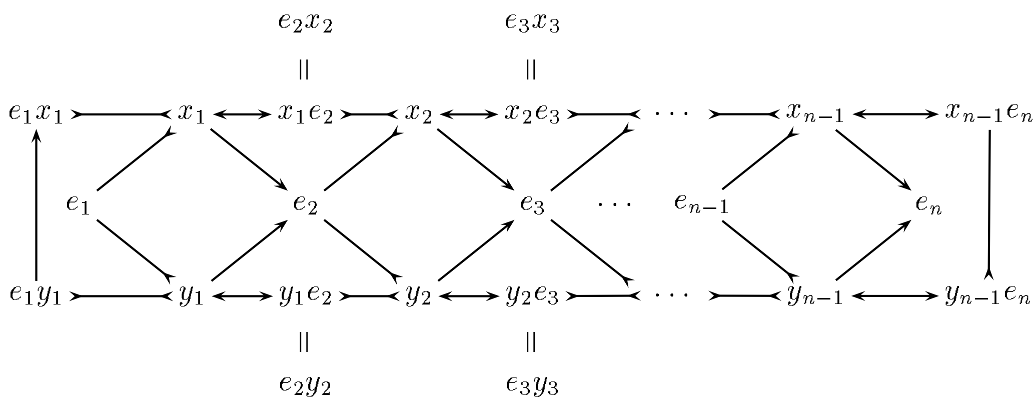
7.2 Sandwich Sets of Sequences

The notion of the sandwich set of a sequence of n idempotents for any $n \geq 2$ was first formulated by Pastijn [26], and generalises the definition of the previous subsection. The terminology here is adapted from [10, 13].

Let $e_1, e_2, \ldots, e_n$ be elements of a biordered set E. We call a sequence $(y_1, \ldots, y_{n-1})$ of elements of E *complete* with respect to $e_1, e_2, \ldots, e_n$ if $e_i \mathrel{-\!\!\!-\!\!\!\prec} y_i \longrightarrow e_{i+1}$ and $y_i e_{i+1} = e_{i+1} y_{i+1}$ for each $i = 1$ to $n-2$. Define $\mathcal{U}(e_1, \ldots, e_n)$ to be the set

$$\{(y_1, \ldots, y_{n-1}) \mid (y_1, \ldots, y_{n-1}) \text{ is complete with respect to } e_1, \ldots, e_n\}.$$

Consider the following diagram:



We say the complete sequence $(x_1, \dots, x_{n-1})$ *lies above* the complete sequence $(y_1, \dots, y_{n-1})$ because of the direction of the left and right arrows at either end of the diagram. The relation of lying above is a pre-order. This observation motivates the following definition.

The *sandwich set* $\mathcal{S}(e_1, \dots, e_n)$ of the elements $e_1, \dots, e_n$ is the set of all $(x_1, \dots, x_{n-1}) \in \mathcal{U}(e_1, \dots, e_n)$ such that $e_1y_1 \longrightarrow e_1x_1$ and $y_{n-1}e_n \rightarrowtail x_{n-1}e_n$ for all $(y_1, \dots, y_{n-1}) \in \mathcal{U}(e_1, \dots, e_n)$. That is, it is the maximal layer with respect to the pre-order defined by the above diagram. Note that in the case where $n = 2$, this definition reduces to the one given in the previous section.

The following result may be proved by a method similar to the proof of Proposition 7.1, and the details are omitted.

Proposition 7.3. Let S be a semigroup, $e_1, \dots, e_n \in E(S)$, and suppose that $e_1 \dots e_n$ is regular with inverse a. Then $(x_1, \dots, x_{n-1}) \in \mathcal{S}(e_1, \dots, e_n)$ where, for $i = 1, \dots, n-2$,

$$x_i \;=\; e_{i+1} \dots e_n \, a \, e_1 \dots e_i.$$

Further, if $y_i = x_i$ or 1 (the formal 1 of S^1), for $i = 1, \dots, n-1$, then

$$e_1 \dots e_n \;=\; e_1 \, y_1 \, e_2 \, y_2 \dots e_{n-1} \, y_{n-1} \, e_n.$$

The last claim of the proposition says that we may 'sandwich' as many of $x_1, \ldots, x_{n-1}$ as we like between the corresponding $e_1, \ldots e_n$ without changing the product $e_1 \ldots e_n$. An application of this appears below in the proof of Theorem 7.5. The formula for x_i was discovered, for general n, by FitzGerald [17], who used the technique to prove the non-trivial result that the idempotent-generated subsemigroup of a regular semigroup is regular. The reader may also note that when $n = 2$, $e_1 = e$, $e_2 = f$, the formula for x_1 becomes $x_1 = f(ef)'e$, which was used in the proof of Proposition 7.1.

Corollary 7.4. Let E be a non-empty biordered set. The following are equivalent:

(1) $E = E(S)$ for some regular semigroup S.

(2) $(\forall e, f \in E) \quad \mathcal{S}(e, f) \neq \emptyset$.

(3) $(\forall n \geq 2)\ (\forall e_1, \ldots, e_n \in E) \quad \mathcal{S}(e_1, \ldots, e_n) \neq \emptyset$.

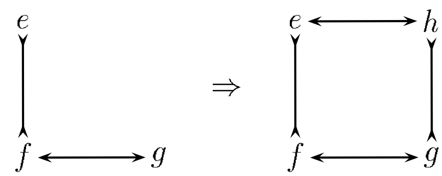
Proof. Clearly (3) implies (2). That (2) implies (1) follows by Theorem 7.2, and (1) implies (3) by the previous proposition. □

The equivalence of (2) and (3) in this corollary was observed by Pastijn [26, 3.10]. The result has been stated in this way to place in context the more general discussion of idempotents of eventually regular semigroups, defined and covered in Section 10 below.

7.3 Unions of Groups

Recall that a semigroup S is called a *union of groups* if each element lies in some subgroup of S. From Propositions 5.5 and 5.6, S must be a disjoint union of maximal subgroups, each corresponding to an $\mathcal{H}$-class of S. The following idea is due to Hall [18], though the terminology comes from the later paper of Clifford [6].

We say that a set of idempotents E is *solid* if for all e, f, $g \in E$,



for some $h \in E$.

Building on the previous discussion of regularity in semigroups and biordered sets, we have the following result about unions of groups due to Hall [18]. Here we follow Easdown [13] who reformulated the statement and proof in the language of biordered sets.

Theorem 7.5. Let $S = \langle E \rangle$ be a regular semigroup, where $E = E(S)$ is the set of idempotents of S. Then S is a union of groups if and only if E is solid.

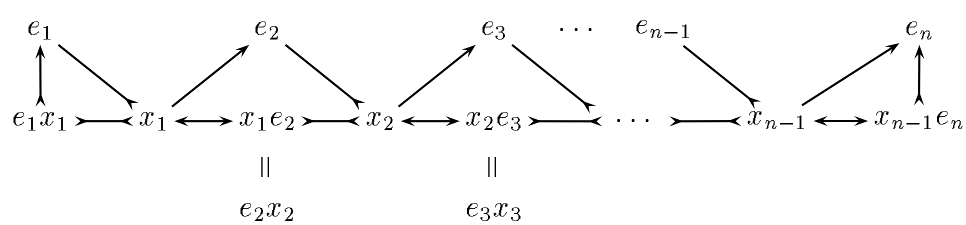
Proof. Let $e, f, g \in E$ such that $e \succ\!\!\!-\!\!\!\prec f \longleftrightarrow g$. Then $e\mathcal{L}f\mathcal{R}g$, so $e\mathcal{R}s\mathcal{L}g$ for some element $s \in S$, since $\mathcal{L}$ and $\mathcal{R}$ commute by Proposition 5.2. Each $\mathcal{H}$-class is a group, so H_s must contain some idempotent h, by Proposition 5.5. Thus $e\mathcal{R}h\mathcal{L}g$, and so $e \longleftrightarrow h \succ\!\!\!-\!\!\!\prec g$, proving the first direction.

Now suppose E is solid, and take an arbitrary element $e_1 e_2 \ldots e_n \in \langle E \rangle$, where the $e_1, \ldots, e_n \in E$. We need only to locate an idempotent x in $H_{e_1 e_2 \ldots e_n}$ to prove the reverse direction, since by Proposition 5.5 this implies that an arbitrary $\mathcal{H}$-class must be a group, whence S is a union of groups.

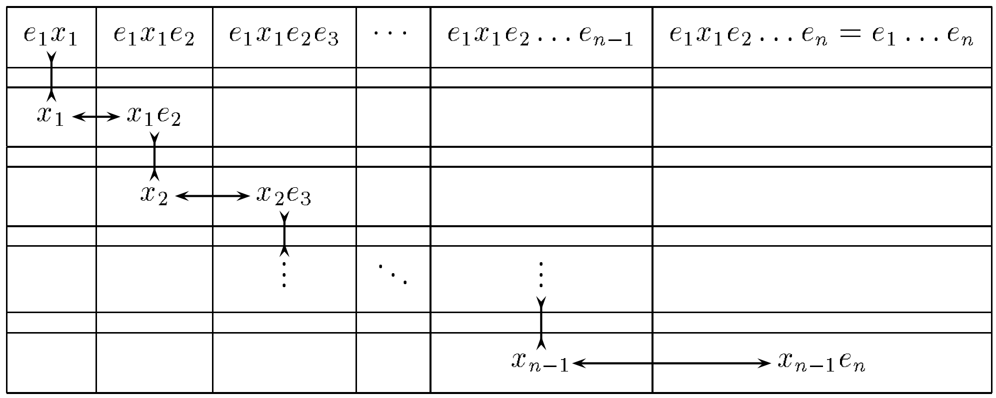
The semigroup S is regular, so we let a be an inverse of $e_1 e_2 \ldots e_n$, and for $i = 1$ to $n-1$, we consider the idempotent

$$x_i = e_{i+1} e_{i+2} \ldots e_n a e_1 e_2 \ldots e_i.$$

By Proposition 7.3, $(x_1, \ldots, x_{n-1}) \in \mathcal{S}(e_1, \ldots, e_n)$ and $e_1 \ldots e_n \; = \; e_1 \, x_1 \, e_2 \ldots e_n$. This gives the following diagram:



If we construct an eggbox diagram, Green's Lemma guarantees that, for $i = 1, \ldots, n-1$, right multiplication by e_i gives us an $\mathcal{R}$-class preserving bijection between the $\mathcal{L}$-class (column) of x_i and the $\mathcal{L}$-class of $x_i\,e_{i+1}$.



By induction, we get that for $i = 2, \ldots, n$,

$$e_1\,x_1 \;\mathcal{R}\; e_1\,x_1\,e_2\ldots e_i \;\mathcal{L}\; x_{i-1}\,e_i.$$

In particular,

$$e_1\,x_1 \;\mathcal{R}\; e_1\,x_1\,e_2\ldots e_n \;=\; e_1\ldots e_n \;\mathcal{L}\; x_{n-1}\,e_n.$$

Using the solidity property, we can locate idempotents in every cell of the diagram. In particular there is an idempotent y such that $e_1\,x_1 \longleftrightarrow y \succ\!\!-\!\!\prec x_{n-1}\,e_n$, whence $y \;\mathcal{R}\; e_1\,x_1 \;\mathcal{R}\; e_1\ldots e_n \;\mathcal{L}\; x_{n-1}\,e_n \;\mathcal{L}\; y$, yielding $y \;\mathcal{H}\; e_1\ldots e_n$. Hence S is a union of groups, proving the reverse implication. $\Box$

8 Representing Biordered Sets Using Partial Transformations

In order to pass beyond the class of regular semigroups, Easdown [9] discovered a faithful representation of an arbitrary biordered set by a semigroup of pairs of transformations (analogous to the representation of a regular semigroup due to Lallement referred to in the first paragraph of Section 7). This representation is described in this section and then applied in subsequent sections to characterise the biordered sets of some important classes of semigroups.

We call a mapping of biordered sets $\theta : E \to F$ a *morphism* if it preserves multiplication, that is

$$(e,f) \in D_E \quad \Rightarrow \quad (e\theta, f\theta) \in D_F \text{ and } (ef)\theta = e\theta f\theta.$$

If θ reverses the order of multiplication, it is referred to as an *anti-morphism*. A biordered set *representation* is any morphism from an abstract biordered set into some other biordered set, typically the biordered set of some known semigroup. Further, we call θ an *isomorphism* if it is a bijection and the inverse mapping θ^{-1} is also a morphism of biordered sets, in which case we say that θ is a *faithful* representation and write $E \cong F$. Note that, since the multiplication of E is partial, it is not automatic that if θ is a bijective morphism then θ^{-1} is also.

8.1 Definitions of the ρ, λ and ϕ mappings

Following Easdown and Hall [15], let E be a biordered set. Write $\mathcal{L} = \rightarrowtail\!\!\prec$, and $\mathcal{R} = \longleftrightarrow$, and let ∞ be a new symbol not lying in $E/\mathcal{L}$ or $E/\mathcal{R}$. Define the maps

$$\rho : E \longrightarrow \mathcal{T}(E/\mathcal{L} \cup \{\infty\}) \qquad \text{and} \qquad \lambda : E \longrightarrow \mathcal{T}(E/\mathcal{R} \cup \{\infty\})$$

by

$$\rho : \quad e \longmapsto \rho_e : \begin{cases} L \longmapsto \begin{cases} L_{xe} & \text{if } x \longrightarrow e \text{ for some } x \text{ in } L \\ \infty & \text{otherwise} \end{cases} \\ \infty \longmapsto \infty, \end{cases}$$

$$\lambda: \quad e \longmapsto \lambda_e : \begin{cases} R \longmapsto \begin{cases} R_{ex} & \text{if } x \succ\!\!\!\longrightarrow e \text{ for some } x \text{ in } R \\ \infty & \text{otherwise} \end{cases} \\ \infty \longmapsto \infty. \end{cases}$$

Building on these definitions, we also define the map

$$\begin{aligned} \phi : \quad E \quad &\longrightarrow \quad \mathcal{T}(E/\mathcal{L} \cup \{\infty\}) \times \mathcal{T}^*(E/\mathcal{R} \cup \{\infty\}) \\ e \quad &\longmapsto \quad \phi_e = (\rho_e, {\lambda_e}^*). \end{aligned}$$

Though each ρ_e and λ_e is defined to be a full transformation, one may think of them as partial transformations where the symbol ∞ is interpreted as 'undefined'.

The following theorem is noted in [15] and proved in [9]. The proof involves difficult manipulations involving the biordered set axioms and the details are not given here.

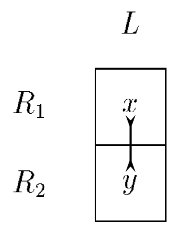
Theorem 8.1. The mapping ϕ defined above is one-one, and preserves and reflects arrows and biordered set products. Thus, as biordered sets, $E \cong E\phi$.

In light of this result, ϕ is a faithful biordered set representation of elements of E by the biordered set of the direct product of a transformation semigroup with the dual of a transformation semigroup. The subsemigroup $\langle E\phi \rangle$ generated by the image of ϕ plays an important role in the remainder of this essay. An example of how a biordered set theoretic property translates through ϕ into a useful property of the semigroup $\langle E\phi \rangle$ is given in the following theorem, the statement of which is extracted from [10, proof of Theorem 9]. The proof is very difficult and is not given here. This theorem will be applied later in Section 11 in characterising the biordered sets of finite semigroups.

Theorem 8.2. Let $e_1, \ldots, e_n \in E$ and suppose $\mathcal{S}(e_1, \ldots, e_n) \neq \emptyset$ and $\phi_{e_1} \ldots \phi_{e_n}$ is idempotent. Then $\phi_{e_1} \ldots \phi_{e_n} = \phi_e$ for some $e \in E$.

8.2 Illustrations

Example 8.3. Let L be a left zero semigroup of two elements. Then L has one $\mathcal{L}$-class and two $\mathcal{R}$-classes, as depicted in the eggbox diagram below.



By direct application of the definitions of the λ and ρ mappings, we have the following calculations:

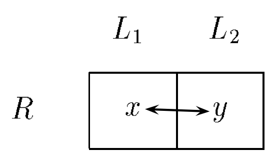
$$\rho_x = \rho_y : \quad L \mapsto L$$

$$\lambda_x : \quad R_1 \mapsto R_1, \quad R_2 \mapsto R_1$$

$$\lambda_y : \quad R_1 \mapsto R_2, \quad R_2 \mapsto R_2$$

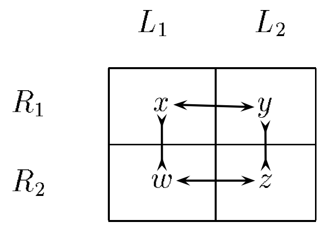
Observe that ρ is a trivial mapping, and that λ is faithful, mapping each element of L to a constant mapping of $\mathcal{R}$-classes. In fact, λ reverses the order of multiplication, since $\lambda_{xy} = \lambda_x = \lambda_y \lambda_x$, and $\lambda_{yx} = \lambda_y = \lambda_x \lambda_y$, producing an anti-isomorphic copy of L. □

Example 8.4. Let R be a right zero semigroup of two elements. Then R has two $\mathcal{L}$-classes and one $\mathcal{R}$-class, as depicted in the eggbox diagram below.



In this example, dual to Example 8.3, λ is trivial and ρ is faithful. This time, ρ preserves the order of multiplication. □

Example 8.5. As noted in Section 1.1, the product of the semigroups L and R described in the previous two examples yields a rectangular band $L \times R$.



Applying the ρ and λ mappings to the biordered set gives the following, where the mappings produced are represented in the two-row format introduced in Example 5.7:

$$\rho_x = \rho_w = \begin{pmatrix} L_1 & L_2 \\ L_1 & L_1 \end{pmatrix}, \qquad \rho_y = \rho_z = \begin{pmatrix} L_1 & L_2 \\ L_2 & L_2 \end{pmatrix}$$

$$\lambda_x = \lambda_y = \begin{pmatrix} R_1 & R_2 \\ R_1 & R_1 \end{pmatrix}, \qquad \lambda_z = \lambda_w = \begin{pmatrix} R_1 & R_2 \\ R_2 & R_2 \end{pmatrix}$$

By identifying $L_i \equiv i \equiv R_i$ for $i = 1, 2$, these yield:

$$\phi_x = \left(\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^* \right) \qquad \phi_y = \left(\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^* \right)$$

$$\phi_w = \left(\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}^* \right) \qquad \phi_z = \left(\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}^* \right),$$

which can be arranged in the following eggbox, noting the characterisations of $\mathcal{L}$ and $\mathcal{R}$ in Example 5.7 in terms of images and kernels of transformations, and also that $\mathcal{L}$ and $\mathcal{R}$ are interchanged by passing to the dual.

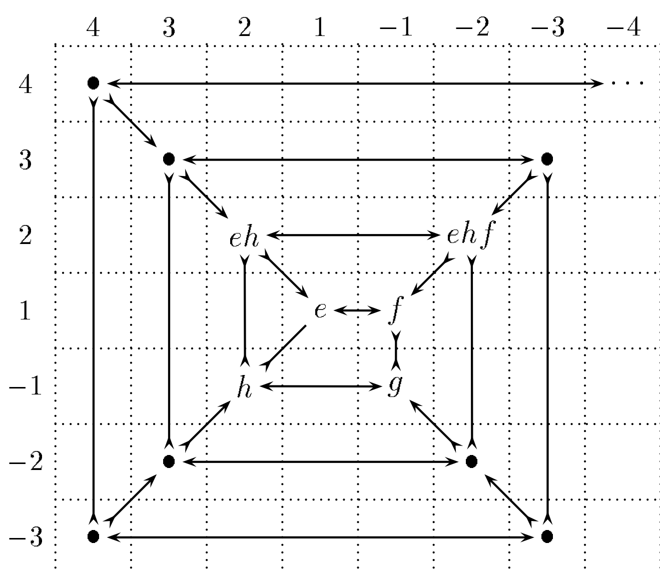
ϕ_x	ϕ_y
ϕ_w	ϕ_z

Thus ϕ recovers the original semigroup from its underlying biordered set, explaining the comment at the end of Example 4.2. $\square$

Example 8.6. We saw in Example 4.1 how any semilattice S may be regarded as a biordered set. For any $e, x \in S$ such that $x \longrightarrow e$, we see $\rho_e : L_x \mapsto L_{xe} = L_x$ since $x \rightarrowtail e$. In S, both $\longleftrightarrow$ and $\rightarrowtail\!\!\!\!\leftarrowtail$ are the identity relation so we may identify L_x and R_x with x for each $x \in S$. Also, $\rho = \lambda$ since $\longrightarrow \; = \; \succ\!\!\!\longrightarrow$. Thus for each $e \in E$, $\rho_e = \lambda_e = \mathrm{id}|_{\{x \mid x \leq e\}}$.

Now consider $e, f, g \in S$ where $g = e \wedge f$. We can calculate $\rho_e \rho_f = \rho_g$ so that $\langle E\rho \rangle$ recovers the semigroup of commuting idempotents of Proposition 2.1. Further, we can also check also that $\mathcal{S}(e, f) = g$, so that the location of products (in this case greatest lower bounds) is determined by sandwich sets. $\square$

Example 8.7. The four spiral biordered set is reproduced below, with $\mathcal{L}$ and $\mathcal{R}$-classes labelled by non-zero integers.



Applying the ρ and λ mappings to the biordered set gives the following, where the mappings produced are given as partial mappings of $\mathbb{Z}\backslash\{0\}$. The integers here refer to the labels of the $\mathcal{L}$-classes in the case of ρ mappings, and $\mathcal{R}$-classes in the case of λ mappings. The symbol ∞ is used to indicate that the mapping is undefined at that point.

$$\rho_e = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ 1 & 2 & 3 & \cdots & 1 & 2 & 3 & \cdots \end{array}\right)$$

$$\lambda_e = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ 1 & 2 & 3 & \cdots & 2 & 3 & 4 & \cdots \end{array}\right)$$

$$\rho_f = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ -1 & -2 & -3 & \cdots & -1 & -2 & -3 & \cdots \end{array}\right)$$

$$\lambda_f = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ 1 & 2 & 3 & \cdots & 1 & 2 & 3 & \cdots \end{array}\right)$$

$$\rho_g = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ \infty & -1 & -2 & \cdots & -1 & -2 & -3 & \cdots \end{array}\right)$$

$$\lambda_g = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ -1 & -2 & -3 & \cdots & -1 & -2 & -3 & \cdots \end{array}\right)$$

$$\rho_h = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ \infty & 2 & 3 & \cdots & 2 & 3 & 4 & \cdots \end{array}\right)$$

$$\lambda_h = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ \infty & -1 & -2 & \cdots & -1 & -2 & -3 & \cdots \end{array}\right)$$

By composition of partial mappings, we have the following:

$$\rho_f\rho_h = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ 2 & 3 & 4 & \cdots & 2 & 3 & 4 & \cdots \end{array} \right)$$

$$\lambda_h\lambda_f = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ \infty & 1 & 2 & \cdots & 1 & 2 & 3 & \cdots \end{array} \right)$$

$$\rho_e\rho_g\rho_e = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ \infty & 1 & 2 & \cdots & \infty & 1 & 2 & \cdots \end{array} \right)$$

$$\lambda_e\lambda_g\lambda_e = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & \cdots & -1 & -2 & -3 & \cdots \\ 2 & 3 & 4 & \cdots & 3 & 4 & 5 & \cdots \end{array} \right)$$

Further compositions yield immediately

$$(\rho_f\rho_h)(\rho_e\rho_g\rho_e) = \rho_e, \qquad (\lambda_e\lambda_g\lambda_e)(\lambda_h\lambda_f) = \lambda_e$$

whence

$$\begin{aligned} (\phi_f\phi_h)(\phi_e\phi_g\phi_e) &= (\rho_f\rho_h\rho_e\rho_g\rho_e,\ {\lambda_f}^*{\lambda_h}^*{\lambda_e}^*{\lambda_g}^*{\lambda_e}^*) \\ &= (\rho_f\rho_h\rho_e\rho_g\rho_e,\ (\lambda_e\lambda_g\lambda_e\lambda_h\lambda_f)^*) \\ &= (\rho_e,\ {\lambda_e}^*) \ = \ \phi_e. \end{aligned}$$

Put $X = \phi_f\phi_h$ and $Y = \phi_e\phi_g\phi_e$, so $XY = \phi_e$. Observe further that

$$X\phi_e \ = \ \phi_e X \ = \ X, \quad Y\phi_e \ = \ \phi_e Y \ = \ Y,$$

so that ϕ_e is the identity element of the monoid B generated by X and Y. Notice also that by direct calculation, $YX \neq \phi_e$. It follows from Lemma 8.8 below that B is isomorphic to the bicyclic semigroup. To see a direct connection with Example 2.4, let $\bar{X}$ and $\bar{Y}$ be the result of modifying X and Y respectively by restricting the domains of the partial mappings to subsets of $\mathbb{Z}^+$. Then

$$\bar{X} = \left(\begin{pmatrix} 1 & 2 & 3 & \cdots \\ 2 & 3 & 4 & \cdots \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots \\ \infty & 1 & 2 & \cdots \end{pmatrix}^* \right)$$

$$\bar{Y} \;=\; \left(\begin{pmatrix} 1 & 2 & 3 & \cdots \\ \infty & 1 & 2 & \cdots \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 2 & 3 & 4 & \cdots \end{pmatrix}^* \right)$$

so that, after renaming the elements of the partially ordered set in Example 2.4 by positive integers, $\bar{X} = (x, x^*)$, and $\bar{Y} = (y, x^*)$. Thus as expected,

$$\bar{X}\bar{Y} \;=\; (xy, y^*x^*) \;=\; (xy, (xy)^*) \;=\; (\mathrm{id}_{\mathbb{Z}^+}, {\mathrm{id}_{\mathbb{Z}^+}}^*)$$

and

$$\bar{Y}\bar{X} \;=\; (\mathrm{id}_{\{z \mid z \geq 2\}}, {\mathrm{id}_{\{z \mid z \geq 2\}}}^*).$$

It turns out that the four spiral semigroup contains four copies of the bicyclic semigroup plus one additional column of elements $\mathcal{L}$-related to e, as suggested by the diagram. Details can be found in [2]. □

The following lemma was used in the above example, and can be found in [7, Lemma 1.31], an abbreviated proof of which we give for completeness.

Lemma 8.8. Let B be a monoid generated by x, y such that $xy = 1$ but $yx \neq 1$. Then $B \cong \langle a, b \mid ab = 1 \rangle_{\mathrm{monoid}}$.

Proof. Suppose the epimorphism induced by $a \mapsto x$, $b \mapsto y$ is not injective. Then $x^i y^j = x^k y^l$ for some $i, j, k, l \geq 0$ such that $(i, j) \neq (k, l)$. It follows quickly that $y^\alpha x^\beta = 1$ for some $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$, whence

$$\begin{aligned} yx \;&=\; x^\alpha y^\alpha y x x^\beta y^\beta \;=\; x^\alpha y y^\alpha x^\beta x y^\beta \\ &=\; x^\alpha y x y^\beta \;=\; x^\alpha y^\beta \qquad\qquad (\text{since } \alpha \geq 1 \text{ or } \beta \geq 1) \\ &=\; x^\alpha y^\alpha x^\beta y^\beta \;=\; 1, \end{aligned}$$

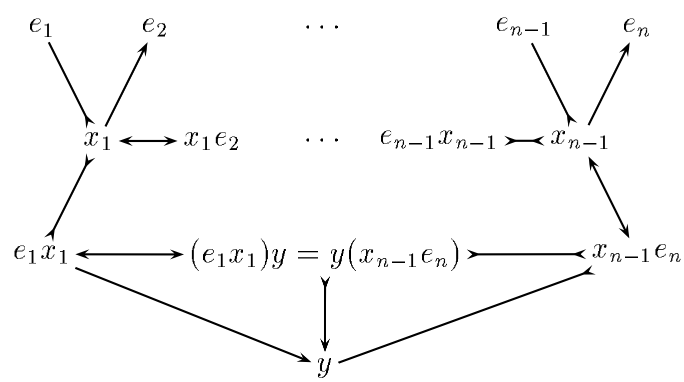
contradicting that $yx \neq 1$. □

9 Solidity and Biordered Sets of Bands

We saw in Section 7 that regularity and solidity together characterise the biordered sets of unions of groups. A union of groups becomes a band if all the groups are trivial. It is natural to ask what additional condition could be imposed on a biordered set E to guarantee that $E = E(B)$ for some band B. In this section we give a more general notion of solidity (equivalent by Proposition 9.1 below to the earlier one in context), but which can be applied to bands and, perhaps surprisingly, also to important subclasses of eventually regular semigroups, discussed in Section 10.

9.1 Solidity

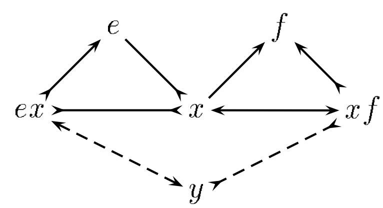
Let E be a biordered set containing $e_1, \ldots, e_n$ and let X be a subset of the set of complete sequences $\mathcal{U}(e_1, \ldots, e_n)$. We say that X is *solid* if there is some $y \in E$ such that for all sequences $(x_1, \ldots, x_{n-1}) \in X$ we have the following relations:



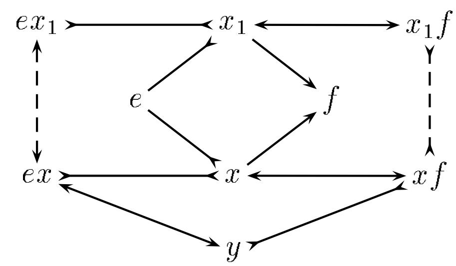
Note that, in this definition, if $X = \emptyset$ then X is vacuously solid.

Proposition 9.1. Let E be a biordered set. Then E is solid (in the sense defined in the paragraph preceding Theorem 7.5) if and only if the sandwich set $\mathcal{S}(e, f)$ is solid for all $e, f \in E$.

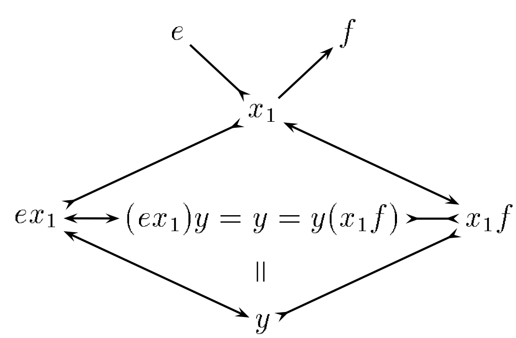
Proof. Suppose E is solid in the earlier sense. Take any $e, f \in E$. If $\mathcal{S}(e,f)$ is empty, then $\mathcal{S}(e,f)$ is vacuously solid. If not, take any $x \in \mathcal{S}(e,f)$. From the definition of complete sequence, we get:



The double left and right arrows at x allow us to use the property of solidity to identify an element y having the relations given by the dashed lines. Let $x_1 \in \mathcal{S}(e,f)$. Thus:

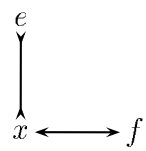


The dashed arrows follow from the definition of membership of the sandwich set. By transitivity of arrows, we have $ex_1 \longleftrightarrow y$ and $x_1 f \rightarrowtail y$. Then we have the following diagram:



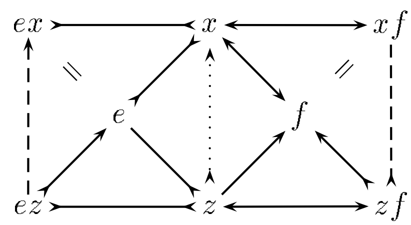
This proves $\mathcal{S}(e,f)$ is a solid subset of $\mathcal{U}(e,f)$.

Now for the converse, suppose $\mathcal{S}(e,f)$ is solid for all $e, f \in E$, and that we have

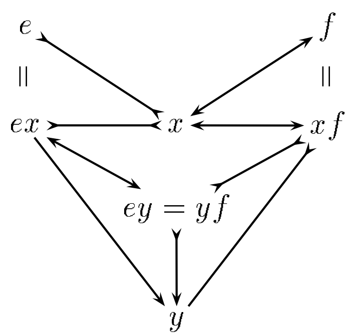


for some x, e, $f \in E$.

It follows that $x \in \mathcal{U}(e, f)$, and in fact $x \in \mathcal{S}(e, f)$ since for any other $z \in \mathcal{U}(e, f)$ we get the following diagram, where the dotted double arrow follows by transitivity, and the dashed arrows follow from axioms (B3*) and (B3).

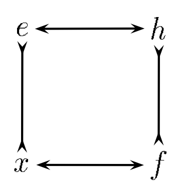


By assumption, $\mathcal{S}(e, f)$ is solid. Hence,



for some $y \in E$.

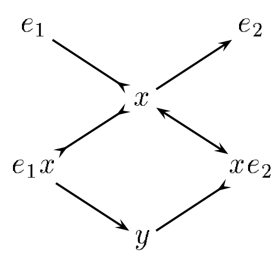
Putting $h = ey = yf$, we immediately have:



Thus E is solid in the earlier sense, proving the reverse direction. $\square$

9.2 Biordered sets of bands

Let B be a band and let $E = E(B)$ be the biordered set of B. Note that whilst $E = B$ as a set, only the band products that arise due to the presence of arrows are retained in the information of the biordered set. Since B is a regular semigroup, E is a regular biordered set by Proposition 7.1. Let $e_1, e_2 \in B$ and put $y = e_1 e_2$, the band product. Let $x \in \mathcal{U}(e_1, e_2)$. Then $y(e_1 x) = e_1 e_2 e_1 x = e_1 e_2 e_1 e_2 x = e_1 e_2 x = e_1 x$ and dually $(x e_2) y = x e_2$, so that the following diagram holds:

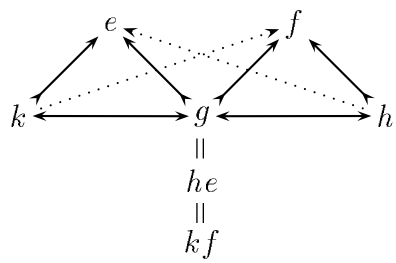


Moreover, $(e_1 x) y = e_1 x e_1 e_2 = e_1 e_2 x e_2 = y(x e_2)$. This proves that $\mathcal{U}(e_1, e_2)$ is solid in the sense of the definition of Section 9.1. We have verified the 'only if' part of Theorem 9.2 below. To prove the 'if' direction, first observe that if a given biordered set E is regular and each $\mathcal{U}(x, y)$ is solid, then in particular, each $\mathcal{S}(x, y)$ is solid, so by Proposition 9.1 and Theorem 7.5, $E = E(S)$ for some union of groups S. The task then is to show that all the subgroups of S are trivial which, by Proposition 5.6, is equivalent to showing that all the $\mathcal{H}$-classes are singleton sets. We can do this by taking $S = \langle E\phi \rangle$ and calculating with ϕ. The complete proof of this is given in [11].

Theorem 9.2. If E is a biordered set then $E \cong E(B)$ for some band B if and only if E is regular and $\mathcal{U}(x, y)$ is solid for all $x, y \in E$.

As an illustration, we apply this theorem to the following example.

Example 9.3. Let E be the following biordered set (E_1 in Example 4.4 above).



Then, by direct calculation from the definitions,

$$\mathcal{U}(e,f) = \{g,k\} \quad \text{and} \quad \mathcal{S}(e,f) = \{g\}.$$

It is routine to check that $\mathcal{U}(e,f)$ is solid, and further that $\mathcal{S}(x,y) \neq \emptyset$ and $\mathcal{U}(x,y)$ is solid for all $x, y \in E$. Thus E is the biordered set of a band by Theorem 9.2. The only band products missing from the data of E are ef and fe. A routine calculation shows that $\phi_e\phi_f = \phi_f\phi_e = \phi_g$, so that we may take the band products ef and fe to be g. In fact, ρ is also a faithful representation of E. $\square$

10 Eventually Regular Semigroups

We say that an element s of a semigroup S is *eventually regular* if it can be powered up to some regular element, that is, for some positive integer n, s^n has an inverse in S. If all elements of S are eventually regular, S is called an *eventually regular semigroup*. We call S *group-bound* if every element powers up to lie in a subgroup of S, so that it has a group inverse, and *periodic* if every element powers up to become idempotent. Certainly all finite semigroups are periodic, all periodic semigroups are group-bound, and all group-bound semigroups are eventually regular. However, these classes of semigroup are all distinct. The infinite cyclic group is clearly not periodic, and an infinite direct sum of non-trivial finite semigroups is periodic but not finite. The bicyclic semigroup, introduced in Example 2.4, is (eventually) regular but not group-bound since no power of the element x falls in the $\mathcal{H}$-class of an idempotent.

The notion of eventually regular was introduced by Edwards [16], who noticed that a number of results proved separately for different subclasses, such as finite or regular semigroups, could be unified and given a single proof for the entire class of eventually regular semigroups. Easdown and Hall proved in [15] that if a biordered set E comes from an eventually regular semigroup then $E \cong E(\langle E\phi \rangle)$. This fact was exploited by Easdown in [10] to characterise the abstract biordered sets which arise from particular subclasses of eventually regular semigroups.

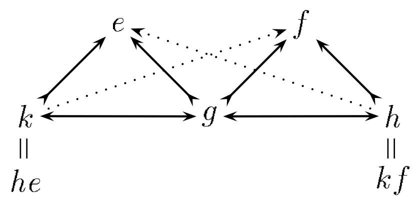
There is an interesting parallel between the condition which characterises biordered sets of regular semigroups, and that which characterises eventually regular semigroups. Whereas we have non-empty sandwich sets of pairs of idempotents in the regular case, for an eventually regular semigroup, arbitrary-sized sequences of idempotents 'eventually' have non-empty sandwich sets, in a sense made precise by the following theorem due to Easdown. For what follows, we use $(e_1, \dots, e_n)^i$ to denote

$$(\underbrace{\underbrace{e_1, \dots, e_n}, \underbrace{e_1, \dots, e_n}, \dots, \underbrace{e_1, \dots, e_n}}_{i \text{ copies}}).$$

Theorem 10.1. A biordered set E comes from an eventually regular semigroup if and only if for each $n \in \mathbb{Z}^+$ and for all $e_1, \dots e_n \in E$ there exists some $i \in \mathbb{Z}^+$ such that $\mathcal{S}(e_1, \dots, e_n)^i$ is non-empty.

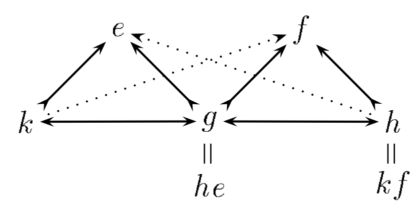
Proof. If $E = E(S)$ for some eventually regular semigroup S, and $e_1, \ldots, e_n \in E$, then $(e_1 \ldots e_n)^i$ is regular for some i so, by Proposition 7.3, $\mathcal{S}(e_1, \ldots, e_n)^i \neq \emptyset$. Conversely, if the condition on the sandwich sets holds, then Easdown proves in [10] that $E \cong E\phi = E(\langle E\phi \rangle)$ and that $\langle E\phi \rangle$ is eventually regular. The details are long and difficult, so not given here. □

Example 10.2. Let E be the following biordered set (E_2 in Example 4.4 above).



By inspection, $\mathcal{U}(e, f) = \{g, k\}, \mathcal{U}(e, f)^2 = \{(g, g, g), (k, h, k)\}$ and, for $i \geq 3$, $\mathcal{U}(e, f)^i = \{(g, \ldots, g), (k, h, \ldots, k, h, k)\}$. But there is no arrow of the form $gf = g \rightarrowtail h = kf$ or $kf = h \rightarrowtail g = gf$ so, by definition, $\mathcal{S}(e, f)^i = \emptyset$ for all $i \geq 1$. By Theorem 10.1, E is not the biordered set of any eventually regular semigroup. A simple calculation reveals that $\langle E\phi \rangle$ has idempotents other than $E\phi$, so $E \ncong E(\langle E\phi \rangle)$. Nevertheless E does arise as the biordered set of a semigroup, namely the free semigroup on E defined in [12]. □

Example 10.3. Let E be the following biordered set (E_3 in Example 4.4 above).



By inspection, $\mathcal{U}(e, f) = \{g, k\}$ and $\mathcal{S}(e, f) = \emptyset$. By Proposition 7.1, E is not the biordered set of any regular semigroup. However $\mathcal{U}(e, f)^2 = \{g, g, g), (k, h, g)\} = \mathcal{S}(e, f)^2$. It turns out that for all $e_1, \ldots, e_n \in E$, $\mathcal{S}(e_1, \ldots, e_n)^i$ is non-empty for some i, but it is not obvious how to check this directly, even for such a small biordered set. The issue of reducing the checking of infinitely many conditions to a finite subcollection is addressed in Section 11. In this example, an indirect method is to calculate $\langle E\phi \rangle$, observe that $E\phi = E(\langle E\phi \rangle)$, see that $\langle E\phi \rangle$ is finite (so eventually regular), and then invoke the 'only if' part of Theorem 10.1. $\square$

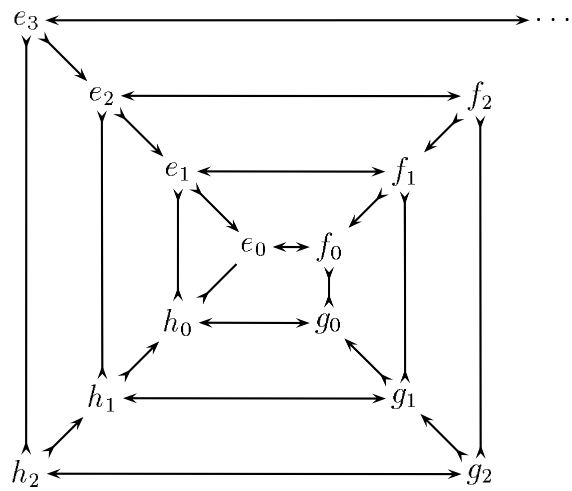
10.1 Group-bound semigroups

We can now use the notion of solidity, introduced in Section 7.3, to draw parallels between the condition characterising biordered sets of unions of groups and that of group-bound semigroups, and between the condition characterising biordered sets of bands and that of periodic semigroups. In both cases, the new results are obtained by considering sufficiently high iterates of sequences of biordered set elements. The results, proved in [10], are quoted here without proof. We simply remark that in each case the necessity of the condition is easily verified, and the proof of sufficiency involves demonstrating relevant properties of the semigroup $\langle E\phi \rangle$.

Theorem 10.4. A biordered set E comes from a group-bound semigroup if and only if for each $n \in \mathbb{Z}^+$, and for all $e_1, \ldots, e_n \in E$, there exists some $i \in \mathbb{Z}^+$ such that $\mathcal{S}(e_1, \ldots, e_n)^i$ is non-empty and solid.

Theorem 10.5. A biordered set E comes from a periodic semigroup if and only if for each $n \in \mathbb{Z}^+$, and for all $e_1, \ldots, e_n \in E$, there exists some $i \in \mathbb{Z}^+$ such that $\mathcal{S}(e_1, \ldots, e_n)^i$ is non-empty $\mathcal{U}(e_1, \ldots, e_n)^i$ and solid.

Example 10.6. Let E be the four spiral biordered set introduced in Example 4.3, labelled as follows:



By inspection, for each $i \geq 1$, $\mathcal{S}(g_0, e_0)^i = \{(f_{i-1}, h_{i-2}, f_{i-2}, \ldots, h_0, f_0\}$ so that $\mathcal{S}(g_0, e_i)^i$ is not solid since there is no $y \in E$ with the property that $g_0 f_{i-1} = g_{i-1} \longleftrightarrow y \rightarrowtail\!\!\!\!\!\!\leftarrowtail e_0 = f_0 e_0$. By Theorem 10.4, E does not arise as the biordered set of any group-bound semigroup. Nevertheless, by inspection, E is regular, so does arise as the biordered set of a regular semigroup, namely the four spiral semigroup defined in Example 4.3 and studied in [2]. □

11 Finite Semigroups

In the sketched proof of the sufficiency of the condition in Theorem 10.1, the representation ϕ (defined in Section 8.1) is used and $E \cong E\phi = E(\langle E\phi \rangle)$. If E is finite then $\mathcal{T}(E/\rightarrowtail \cup \{\infty\}) \times \mathcal{T}^*(E/\longleftrightarrow \cup \{\infty\})$ is also finite, so Theorem 10.1 yields the following characterisation:

Theorem 11.1. A finite biordered set E comes from a finite semigroup if and only if E is finite and for each $n \in \mathbb{Z}^+$ and for all $e_1, \ldots, e_n \in E$ there exists some $i \in \mathbb{Z}^+$ such that $\mathcal{S}(e_1, \ldots, e_n)^i$ is non-empty.

Checking this condition on sandwich sets ostensibly involves looking at infinitely many sequences, something which is non-trivial for even small finite biordered sets. However, an idea due to Easdown reduces this problem to a finite size. Borrowing a result from graph theory known as Ramsey's Theorem, it is shown below that we need only to check the sandwich sets of a finite number of sequences to test whether a given biordered set comes from a finite semigroup.

Theorem 11.2. Let $m > 0$. Then for some $N = N(m) \in \mathbb{Z}^+$, every complete graph on $n \geq N$ vertices whose edges are coloured with m colours contains a monochromatic triangle.

Proof. This is a special case of Ramsey's Theorem [30, 31]. □

We call the integer $N(m)$ in the above theorem a *Ramsey number.*

Theorem 11.3. Let S be a finite semigroup with m elements.[3] Let $N = N(m)$, the m^{th} Ramsey number. If $x_1, \ldots, x_n \in S$ where $n \geq N$ then $x_j x_{j+1} \ldots x_k \in E(S)$ for some $j \leq k$.

[3]This idea first seems to appear in [1], and a variant of it can be found in [28].

Proof. Let $x_1, \ldots, x_n \in S$ where $n \geq N$. Let Γ be the complete graph on vertices $1, \ldots, n$, and let S be the set of colours. Colour the edge from i to j where $i < j$ by the semigroup product $x_i x_{i+1} \ldots x_{j-1}$. Then Γ has a monochromatic triangle with vertices $i < j < k$. Hence $x_i \ldots x_{j-1} = x_j \ldots x_{k-1} = x_i \ldots x_{k-1}$, whence $x_i \ldots x_{j-1}$ is idempotent. □

Theorem 11.4. Let E be a finite biordered set with l $\succ\!\!-\!\!\prec$-classes and r $\longleftrightarrow$-classes. Put

$$m = \max\{l, r\}, \qquad N = N(q),$$

where q is the size of the set $\mathcal{T}(E/\succ\!\!-\!\!\prec \cup \{\infty\}) \times \mathcal{T}^*(E/\longleftrightarrow \cup \{\infty\})$, and $N(q)$ is Ramsey's number. Then E is the biordered set of some finite semigroup if and only if

$$(\forall n = 1, \ldots, 2N)\,(\forall e_1, \ldots, e_n \in E) \quad \mathcal{S}(e_1, \ldots, e_n)^m \neq \emptyset.$$

Proof. For the forward direction, the reader is referred to [10, page 502]. For the converse, suppose the condition of the theorem holds. Since E is finite, $\langle E\phi \rangle$ is finite. Because $E \cong E\phi$ by Theorem 8.1, it remains to show $E\phi = E(\langle E\phi \rangle)$. Let $e_1, \ldots, e_n \in E$ such that $\phi_{e_1} \ldots \phi_{e_n} \in E(\langle E\phi \rangle)$.

Case (i): Suppose $n \leq 2N$. Then, by the condition we have assumed, $\mathcal{S}(e_1, \ldots, e_n)^m \neq \emptyset$ so by Theorem 8.2, $\phi_{e_1}, \ldots, \phi_{e_n} = (\phi_{e_1}, \ldots, \phi_{e_n})^m \in E\phi$, and we are done. This starts an induction.

Case (ii): Suppose $n > 2N$. Let $\psi_i = \phi_{e_{2i-1}} \phi_{e_{2i}}$ for $i = 1, \ldots, N$. By Theorem 11.3, $\psi_j \ldots \psi_k$ is idempotent for some j, k where $1 \leq j \leq k \leq N$. But $\psi_j \ldots \psi_k = \phi_{e_{2j-1}} \ldots \phi_{e_{2k}}$ contains $2(k-j) \leq 2N$ multiplicands on the right-hand side, and so lies in $E\phi$ by Case (i). Thus, for some $e \in E$, $\phi_{e_1} \ldots \phi_{e_n} = \phi_{e_1} \ldots \phi_{e_{2j-2}} \phi_e \phi_{2j+1} \ldots \phi_{e_n}$, where the right-hand side contains strictly less multiplicands than the left-hand side. By an inductive hypothesis applied to the right-hand side, $\phi_{e_1} \ldots \phi_{e_n} \in E\phi$. By induction, $E(\langle E\phi \rangle) = E\phi$, and the converse direction of the theorem is proved. □

Thus we have found a finite process for characterising the biordered sets of finite semigroups, the final class considered here.

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